AN IMPACT MODEL FOR THE INDUSTRIAL CAM-FOLLOWER SYSTEM:
SIMULATION AND EXPERIMENT

By:

Vasin Paradorn

A Thesis

Submitted to the Faculty

of

WORCESTER POLYTECHNIC INSTITUTE

In partial fulfillment of the requirements for the

Degree of Master of Science

in

Mechanical Engineering

by:

__________________________________________
Vasin Paradorn
October 11th, 2007

APPROVED:

__________________________________________
Professor Robert L. Norton, Major Advisor

__________________________________________
Professor Zhikun Hou, Thesis Committee Member

__________________________________________
Professor John M. Sullivan, Thesis Committee Member

__________________________________________
Professor Cosme Furlong, Graduate Committee Member
ABSTRACT

Automatic assembly machines have many cam-driven linkages that provide motion to tooling. Newer machines are typically designed to operate at higher speeds and may need to handle products with small and delicate features that must be assembled precisely every time. In order to design a good tooling mechanism linkage, the dynamic behavior of the components must be considered; this includes both the gross kinematic motion and self-induced vibration motion.

Current simulations of cam-follower system dynamics correlate poorly to the actual dynamic behavior because they ignore two events common in these machines: impact and over-travel. A new dynamic model was developed with these events. From this model, an insight into proper design of systems with deliberate impact was developed through computer modeling.

To attain more precise representations of these automatic assembly machines, a simplified industrial cam-follower system model was constructed in SolidWorks CAD software. A two-mass, single-degree-of-freedom dynamic model was created in Simulink, a dynamic modeling tool, and validated by comparing to the model results from the cam design program, DYNACAM. After the model was validated, a controlled impact and over-travel mechanism was designed, manufactured, and assembled to a simplified industrial cam-follower system, the Cam Dynamic Test Machine (CDTM). Then, a new three-mass, two-degree-of-freedom dynamic model was created. Once the model was simulated, it was found that the magnitude and the frequency of the vibration, in acceleration comparison, of the dynamic model matched with the experimental results fairly well. The two maximum underestimation errors, which occurred where the two bodies collided, were found to be 119 m/s² or 45% and 41 m/s² or 30%. With the exception of these two impacts, the simulated results predicted the output with reasonable accuracy. At the same time, the maximum simulated impact force overestimated the maximum experimental impact force by 2 lbf or 1.3%.

By using this three-mass, two-DOF impact model, machine design engineers will be able to simulate and predict the behavior of the assembly machines prior to manufacturing. If the results found through the model are determined to be unsatisfactory, modifications to the design can be made and the simulation rerun until an acceptable design is obtained.
ACKNOWLEDGEMENT

I would like to thank the Gillette Company and the Gillette Project Center at Worcester Polytechnic Institute for funding this research. For without them the project would not have been realized.

I would like to express my sincere gratitude and appreciation to my advisor, Professor Robert L. Norton, for his time, guidance, support, and most importantly his patience over the past several years.

I am very grateful to the members of my Thesis Committee, Professor Cosme Furlong-Vazquez, Professor Zhikun Hou, and Professor John M. Sullivan, for their time and assistance in this work, and for guiding me through various courses at WPI.

My thanks go to the WPI Mechanical Engineering department and graduate committee for supporting me with a teaching assistant position during my studies at WPI. Also, the people who made the Mechanical Engineering department feel like home, Barbara Furlman, Barbara Edilberti, and Pam St Louis from ME department office.

My special thanks go to Mr. Sia Najafi for choosing me as his TA for two years, encouragement, and for generously helping me by providing computer facilities and support.

I would like to thank my friends Shilpa Jacobie, Edyta Soltan, Randy Robinson, and Irene Gouverneur for their friendship and support while I was working on this thesis. Thanks also go to my friends and colleagues Adriana Hera and Appu Thomas for all their help, and Elizabeth Norgard for reading my thesis.

I am very grateful to my parents, Klai and Pikul Paradorn, brother, Vachara Paradorn, and sister, Vacharaporn Paradorn, for their unconditional love and continuous support.
EXECUTIVE SUMMARY

Automatic assembly machines have many cam-driven linkages that provide motion to tooling. Newer machines are typically designed to operate at higher speeds and may need to handle products with small and delicate features that must be assembled precisely every time. In order to design a good tooling mechanism linkage, the dynamic behavior of the components must be considered; this includes both the gross kinematic motion and self-induced vibration motion.

Current simulations of cam-follower system dynamics correlate poorly to the actual dynamic behavior because they ignore two events common in these machines: impact and over-travel. A new dynamic model was developed with these events. From this model, an insight into proper design of systems with deliberate impact was developed through computer modeling.

To attain more precise representations of these automatic assembly machines, a simplified industrial cam-follower system model was constructed in SolidWorks CAD software. A two-mass, single-degree-of-freedom dynamic model was created in Simulink, a dynamic modeling tool, and validated by comparing to the model results from the cam design program, DYNACAM. After the model was validated, a controlled impact and over-travel mechanism was designed, manufactured, and assembled to a simplified industrial cam-follower system, the Cam Dynamic Test Machine (CDTM). Then, a new three-mass, two-degree-of-freedom dynamic model was created. Dynamic modeling techniques were used to determine the lumped masses of the CDTM. Their stiffness constants and damping coefficients were calculated through either finite element analysis or approximation. Investigation of the best impact force approximation was done prior to finalization of the dynamic model. Once the best impact force approximation was determined, a new dynamic model was fully developed. The experimental data obtained was used to validate the dynamic model with impact and over-travel.

Once the simple, two-mass, single-degree-of-freedom model without impact was correlated with the result from DYNACAM, a three-mass, two-degree-of-freedom model with impact was developed from it. The weights were calculated to be 16.351 lb, 1.638...
lb, and 0.3854 lb for $m_1$, $m_2$, and $m_3$, respectively. Through finite element analysis, stiffness constants $k_{01}$, $k_{12}$, $k_{23}$ push and pull, and $k_{03}$ were determined to be 103,873 lb/in, 9,051 lb/in, 8,144 lb/in and 219 lb/in, and 49,094 lb/in, respectively. Ray C. Johnson’s common velocity approach for determining impact force was determined to be more accurate than the energy method. While the energy method underestimated the impact force from 35% to 40%, Johnson’s method overestimated these same impact forces by 10% to 25%. Thus, Johnson’s method was employed and the three-mass two-DOF impact model was finalized. Once the model was simulated, it was found that the magnitude and the frequency of the vibration, in acceleration comparison, of the dynamic model matched with the experimental results fairly well. The two maximum underestimation errors, which occurred where the two bodies collided, were found to be 119 m/s$^2$ or 45% and 41 m/s$^2$ or 30%. With the exception of these two impacts, the simulated results predicted the output with reasonable accuracy. At the same time, the maximum simulated impact force overestimated the maximum experimental impact force by 2 lbf or 1.3%.

By using this three-mass, two-DOF impact model, machine design engineers will be able to simulate and predict the behavior of the assembly machines prior to manufacturing. If the results found through the model are determined to be unsatisfactory, modifications to the design can be made and the simulation rerun until an acceptable design is obtained.
# Table of Contents

1 INTRODUCTION  ...................................................................................................... 1
2 GOAL ......................................................................................................................... 3
3 LITERATURE REVIEW ........................................................................................... 4  
   3.1 Dynamic Modeling ............................................................................................. 4  
      3.1.1 Single degree-of-freedom model (SDOF) .................................................. 5  
         3.1.1.1 One-mass dynamic models ............................................................... 5  
         3.1.1.2 Two-mass dynamic models ............................................................... 8  
      3.1.2 Multiple degree-of-freedom model (MDOF) ........................................... 9  
3.2 Impact Modeling ............................................................................................... 12  
3.3 Impact Solvers .................................................................................................. 14  
   3.3.1 Energy methods for impact modeling ....................................................... 15  
   3.3.2 Deflection and correction factor approach for impact modeling .......... 16  
   3.3.3 The common velocity approach for impact modeling ......................... 19  
   3.3.4 The wave method for impact modeling ................................................... 22  
4 TEST APPARATUS ................................................................................................. 26  
   4.1 Existing CDTM ............................................................................................... 26  
   4.2 Redesigned CDTM with Impact and Over-travel ....................................... 30  
5 MODELING OF CDTM ........................................................................................... 37  
   5.1 Universal Schematic and Free Body Diagram ............................................. 37  
   5.2 No Contact ...................................................................................................... 38  
   5.3 Initial Contact: Impact ................................................................................... 39  
   5.4 Over-Travel .................................................................................................... 40  
   5.5 Universal Equations of Motion .................................................................... 41  
6 DETERMINING THE PARAMETERS OF THE CDTM ....................................... 43  
   6.1 Lumped Masses Determination .................................................................. 43  
   6.2 Lumped stiffness constants determination .................................................. 46  
7 SOLVING THE 3-MASS 2 DOF DYNAMIC MODEL .......................................... 50  
   7.1 Solution Approaches ..................................................................................... 50  
      7.1.1 Block diagram: Matlab and Simulink ................................................. 50  
   7.2 Solvers .......................................................................................................... 51  
      7.2.1 Stiff and non-stiff systems .................................................................... 51  
      7.2.2 Available solvers .................................................................................. 52  
   7.3 Simulink: 3-Mass MDOF Model ................................................................. 55  
8 RESULTS ................................................................................................................. 61  
   8.1 Simulink Results ........................................................................................... 61  
   8.2 Experimental Results with Impact and Over-travel ................................ 68  
   8.3 Experimental and Simulated Results Comparisons .................................... 74  
   8.4 Simulated No Impact and Impact Comparisons ......................................... 77  
9 SUMMARY AND CONCLUSIONS ....................................................................... 82  
10 RECOMMENDATIONS .................................................................................. 83  
REFERENCES ................................................................................................................. 84  
Appendix A: Dynamic Modeling Techniques ............................................................ 86  
   Mass ...................................................................................................................... 86  
   Spring rate ......................................................................................................... 87  
   Damping .............................................................................................................. 88
Combining the parameters ........................................................................................................... 89
Lever and gear ratio .................................................................................................................. 89
Appendix B: Simulink vs. DYNACAM comparison ................................................................. 94
  Creation of a 2-mass SDOF model ......................................................................................... 94
  Simulink: 2-Mass SDOF Model .......................................................................................... 95
  Validation of Simulink for non-impact model with DYNACAM ........................................ 98
Appendix C: Impact Force Determination ............................................................................. 102
  Calculations of the Impact Parameters .............................................................................. 102
  Validation of Impact Force Approximation: Ball Drop Experiment .................................. 103
Appendix D: Lumped Masses Calculation ............................................................................ 113
Appendix E: Stiffness Constants Calculation ........................................................................ 118
Appendix F: Lumped Stiffness Constants Calculation .......................................................... 140
Appendix G: Impact and Over-Travel Engineering Drawings ............................................. 142
List of Figures

Figure 3.1 - Overhead valve linkage (Barkan 1953)................................. 5
Figure 3.2 - Simplified Valve Train One-Mass Model (Barkan 1953).................. 6
Figure 3.3 - Simplified Valve-Train 2-Mass 1-DOF Model (Dresner & Barkan 1995)........ 7
Figure 3.4 - Simplified 2-Mass 2-DOF Model (Chen et al. 1975)......................... 10
Figure 3.5 - Simplified 3-Mass 2-DOF Model (Norton et al. 2002)......................... 11
Figure 3.6 - Force vs. Deflection (Burr 1982, p. 591).......................... 15
Figure 3.7 - Striking Impact – Vertical Fall (Burr 1982, pp. 591).................... 16
Figure 3.8 - Deflection due to Striking Impact (Burr 1982, pp. 593)........... 17
Figure 3.9 - Common Velocity Linear - Spherical Members.......................... 20
Figure 3.10 - Wave Method Notations (Burr 1982, pp. 596)..................... 22
Figure 3.11 - Horizontal Striking Impact (Burr 1982, pp. 599)............... 23
Figure 4.1 - Overview of the Original CDTM........................................ 26
Figure 4.2 - Input Function: Displacement vs. Cam Angle.................... 28
Figure 4.3 - Input Function: Velocity vs. Cam Angle.......................... 28
Figure 4.4 - Input Function: Acceleration vs. Cam Angle.................... 28
Figure 4.5 - Original Dimensioned CDTM with Sensors......................... 29
Figure 4.6 - Isometric View of the Impact and Over-Travel Mechanism........ 30
Figure 4.7 - Overview of the Impact and Over-Travel Mechanism.............. 31
Figure 4.8 - Impact Mechanism Components....................................... 32
Figure 4.9 – Exploded View: Over-Travel Mechanism Components........... 33
Figure 4.10 – Sectioned View: Over-Travel Mechanism and Force vs. Deflection Plots 34
Figure 4.11 - Final Dimensioned CDTM with Sensors (Parts Hidden)........ 35
Figure 5.1 - Universal Diagram of CDTM ........................................... 37
Figure 5.2 - Diagram of CDTM: Condition 1 – No Contact................... 38
Figure 5.3 - Diagram of CDTM: Condition 2 – Initial Contact: Impact........ 39
Figure 5.4 - Diagram of CDTM: Condition 3 – Over-Travel.......................... 40
Figure 6.1 - CDTM with Impact and Over-Travel Mass Division ............ 43
Figure 6.2 - First Step Lumped Mass.................................................. 44
Figure 6.3 - Second Step Lumped Mass.................................................. 45
Figure 6.4 - Final Lumped Mass Model.................................................. 45
Figure 6.5 - First Lumped Stiffness Constant Model.............................. 47
Figure 6.6 - Final Lumped Stiffness Constant Model.............................. 48
Figure 6.7 - Simplified CDTM: Industrial 3-Mass 2-DOF with Calculated Parameters.. 49
Figure 7.1 - Simulink's 3-Mass 2-DOF Industrial Model........................... 55
Figure 7.2 - Simulink's Sub-System of 3-Mass 2-DOF Industrial Model........ 56
Figure 7.3 - Sub-Section 1: Inputs.......................................................... 57
Figure 7.4 – Sub-Section 2: Damping Calculation for 3-Mass 2-DOF Model ........ 57
Figure 7.5 - Sub-Section 5: Results........................................................ 58
Figure 7.6 - Sub-Section 3: Force Determinations................................ 58
Figure 7.7 - Sub-Section 4: Equations of Motion................................... 58
Figure 7.8 - Over-Travel Force Calculation for 3-Mass 2-DOF Model........ 59
Figure 7.9 - Impact Force Calculation for 3-Mass 2-DOF Model............... 59
Figure 8.1 - CDTM Equivalent 3-Mass 2-DOF Schematic Diagram........... 61
Figure 8.2 - Simulated Displacement Comparison of $M_2$ and $M_3$............. 62
Figure 8.3 - Impact Mechanism: Over-Travel ................................................................. 63
Figure 8.4 – Impact Mechanism: Initial Contact ............................................................. 63
Figure 8.5 – Impact Mechanism: Zero Force ................................................................. 63
Figure 8.6 - Simulated Velocity Comparison of \( M_2 \) and \( M_3 \) ............................................. 64
Figure 8.7 – Normalized Simulated Velocity of \( M_2 \) after the 2\(^{nd} \) Impact ................ 65
Figure 8.8 - Normalized Simulated Velocity of \( M_3 \) after the 2nd Impact ......................... 65
Figure 8.9 - Simulated Acceleration Comparison of \( m_2 \) and \( m_3 \) ................................. 66
Figure 8.10 - Simulated Impact and Over-travel Force ................................................... 67
Figure 8.11 - Final Dimensioned CDTM with Sensors (Parts Hidden) ............................. 68
Figure 8.12 - Experimental Displacement Data with Impact and Over-travel Events ..... 69
Figure 8.13 - Experimental Velocity Data with Impact and Over-travel Events .............. 70
Figure 8.14 - Experimental Acceleration Data with Impact and Over-travel Events ....... 71
Figure 8.15 - Experimental Force Data with Impact and Over-travel Events .................. 72
Figure 8.16 – Normalized Experimental Force and Experimental Displacement Data .... 73
Figure 8.17 - Experimental vs. Simulated Acceleration ............................................... 74
Figure 8.18 - Simulated Acceleration of Intermediate mass \( (M_2) \) ................................. 75
Figure 8.19 - Experimental Acceleration of Intermediate mass \( (M_2) \) .............................. 75
Figure 8.20 - Experimental vs. Simulated Impact and Over-travel Force ....................... 76
Figure 8.21 - Mass 2 Simulated Displacement: No Impact vs. Impact ......................... 77
Figure 8.22 - Mass 2 Simulated Displacement: No Impact vs. Impact % Difference ...... 78
Figure 8.23 - Mass 2 Simulated Velocity: No Impact vs. Impact ................................. 79
Figure 8.24 - Mass 2 Simulated Velocity: No Impact vs. Impact % Difference .......... 79
Figure 8.25 – Mass 2 Simulated Acceleration: No Impact vs. Impact ........................... 80
Figure 8.26 - Mass 2 Simulated Velocity: No Impact vs. Impact % Difference .......... 81
1 INTRODUCTION

Automatic assembly machines have many cam-driven linkages that provide motion to tooling. Newer machines are typically designed to operate at higher speeds and may need to handle products with small and delicate features that must be assembled precisely every time. In order to design a good tooling mechanism linkage, the dynamic behavior of the components must be considered; this includes both the gross kinematic motion and self-induced vibration motion.

Dynamic models were created to obtain insight into dynamic behavior of the system prior to manufacturing. These models were mathematical tools used to simulate and predict the behavior of physical systems. They contain systems’ properties which are masses, stiffness constants, and damping coefficients.

One widely used model is a simplified, two-mass, single degree of freedom dynamic model of the cam-follower system. Unfortunately, the dynamic model being used is not ideal because it lacks impact and over-travel event and has only one degree of freedom. Therefore, a more sophisticated model must be developed and implemented to correlate better with the actual system. This was accomplished by using an existing dynamic model of cam-follower systems and generating a superior dynamic model capable of simulating and predicting the behavior of the systems with these events.

This superior dynamic model was created in Simulink, a tool for modeling, simulating, and analyzing dynamic systems. The result obtained from the dynamic model was compared to DYNACAM’s. After the dynamic model was validated, impact and over-travel mechanisms were developed with CAD tools such as Pro/Engineer and SolidWorks. Using these CAD packages, it was possible to articulate the machine virtually and use Finite Element Analysis to further analyze the individual parts and the loads that acted on them. Once a feasible design was obtained, the parts were manufactured and assembled onto the CDTM, to replicate the events found in the sponsor’s machines.

Determination of the best impact force approximation was also conducted by comparing experimental to simulated results. After the best method was found, it was implemented into the Simulink model. The dynamic model of the machine created in Simulink consists of three masses, five spring constants, and three damping coefficients.
These properties were determined by CAD software, Finite Element Analysis program, and experimentation.

From this information, a superior dynamic model was created in Simulink with appropriate input values. The model created in Simulink was compared to the experimental results obtained through the use of LVDT, LVT, a piezoelectric accelerometer, a force transducer, and a Digital Signal Analyzer. Once the correlations of these two data were determined to be reasonable, the entire processes were recorded, printed, and presented to the sponsor.
2 GOAL

The objective of this thesis is to create a three-mass, two-degree-of-freedom, dynamic model of a cam follower system with impact loading and over-travel events. This model will allow machine design engineers to predict the dynamic behavior of the system prior to manufacturing and determine whether a newly designed machine meets specifications. The new dynamic model will allow users to input the calculated lumped masses, stiffness constants, and damping coefficients of the new machine as well as use the theoretical displacement, velocity, and acceleration of the cam profile as the forcing function. Since impact force must be included in the model, it is also necessary to include the variables that will be used to determine the impact force such as modulus of elasticity and other properties. After these values are inserted and the model run, the Simulink model will allow the designer to see the simulated results. This information may be used to optimize the new machine to obtain improved performance.
3 LITERATURE REVIEW

3.1 Dynamic Modeling

Dynamic modeling is a mathematical tool that is used to describe the behavior of physical systems. These systems may be represented by single or multiple differential equations and may be a mechanical, electrical, thermal, or any other time-varying system. In this particular case, only dynamic models for mechanical systems are considered.

Every real mechanical system has infinite degrees of freedom. The higher the degree of freedom in the model, the more accurate the simulation will be, at the price of model complexity and computation time. In order to have a reasonable computation time and acceptable results, the model needs to be simplified. This simplification may be done by reducing the degrees of freedom by combining masses, stiffness constants, and damping coefficients. The simplest dynamic model is a single degree of freedom model with one mass, one spring, and one damper. More complex models have multiple degrees of freedom with multiple masses, springs, and dampers. Simplifications of complex models to simple models are shown in the following sections.

The application of dynamic modeling to cam-follower systems was first seen in the automotive industry in 1953 when a single-degree-of-freedom dynamic model was created with good correlation between experimental and simulated data (Barkan 1953). Superior correlation was obtained when a twenty-one degree-of-freedom dynamic model was created for the valve-train system (Seidlitz 1989). The disadvantage of the latter model was a longer modeling and computational time. Other applications included modeling of a robotic arm with impact (Ferretti et al 1998) and modeling of industrial cam-follower systems (Norton et al 2002).

By creating a dynamic model, the designer is able to determine the behavior of a system prior to expensive manufacture, assembly, and testing. If the requirements are not met, appropriate fundamental changes may be made early on in the product cycle to obtain acceptable behavior.
3.1.1 Single degree-of-freedom model (SDOF)

A single degree of freedom (SDOF) model is the simplest dynamic model. An SDOF model can have one or two lumped masses and is typically used as a quick approximation of the dynamic behavior of a system prior to increasing the complexity of the model for a more accurate analysis. The advantages and disadvantages of one-mass and two-mass SDOF models are discussed at the end of this subsection.

3.1.1.1 One-mass dynamic models

One-mass SDOF model is a simplified model used to predict the dynamic behavior of the motion of a system. The application and derivation of one-mass SDOF model was explicitly shown in 1953 by Barkan. Prior to Barkan’s work, there were limited uses of dynamic models in the simulation of mechanical systems in the automotive industry. A dynamic model was developed for the high-speed motion of a cam-actuated engine valve and overhead valve linkage shown in Figure 3.1.

![Figure 3.1 - Overhead valve linkage (Barkan 1953)](image)
To simplify the system shown in Figure 3.1, Barkan divided the valve-train into several concentrated masses, and then relocated the masses to the valve head’s axis of translation using the appropriate lever ratios to create one lumped mass. Once the lumped parameters were obtained, equations of motion were developed. To create the equations of motion, the forces acting on the system were identified. These comprised the spring force, inertia force, linkage compression force, friction force, and gas force. Barkan resolved the spring force into valve spring compression force, valve spring preload force, and the force produced due to the vibration of the springs. Three types of friction were taken into account for the damping, namely coulomb friction, viscous friction proportional to relative velocity, and viscous friction proportional to absolute velocity. The most complex portion of the equation was determined to be the gas force, which occurred when there was a difference in pressures. Barkan excluded spring vibration and gas force from the equations of motion because the spring surge had been determined to be insignificant (Oliver and Mills 1945). Other authors disagreed with the elimination of the spring surge and stated that unacceptable errors may occur (Philips, Schamel, and Meyer 1989). The gas force was very complex to model and would have required experimental data which was not readily available, therefore it was neglected. The equations of motion were created for the simplified one mass model shown in Figure 3.2.

![Figure 3.2 - Simplified Valve Train One-Mass Model (Barkan 1953)](image)
Depending on the cam and follower contact condition, one of the following equations would be calculated using the notation in Figure 3.2.

\[ \ddot{x} + 2 \zeta_1 \omega_{n-1} \dot{x} + \omega_{n-1}^2 x = \frac{1}{M} F(t) \]  
\[ (3.1) \]

\[ \ddot{x} + 2 \zeta_2 \omega_{n-2} \dot{x} + \omega_{n-2}^2 x = \frac{1}{M} F(t) \]  
\[ (3.2) \]

\[ \ddot{x} + 2 \zeta_3 \omega_{n-2} \dot{x} + \omega_{n-2}^2 x = \frac{1}{M} F(t) \]  
\[ (3.3) \]

where \( x, \dot{x}, \ddot{x} \) are the displacement, velocity, and acceleration of \( M \) relative to equivalent cam follower or \( Y - y \), \( \dot{Y} - \dot{y} \), and \( \ddot{Y} - \ddot{y} \), respectively and \( \zeta \) are the critical damping factors.

\[ \zeta_1 = \frac{b + \beta}{2M \omega_{n-1}}, \quad \zeta_2 = \frac{\beta}{2M \omega_{n-2}}, \quad \zeta_3 = \frac{b + \beta}{2M \omega_{n-2}} \]  
\[ (3.4) \]

Equations 3.1, 3.2, 3.3 can be applied when \( \dot{y} > 0 \) (valve opening), \( \dot{y} < 0 \) (valve closing), and \( x < 0 \) (valve jumps), respectively. The variable \( x \) represents the displacement of mass \( M \) with respect to equivalent cam. Barkan’s work proved the validity of a mathematical model when he compared the simulated results with the experimental result.

Other works that followed tried to improve upon different aspects of the modeling in attempt to increase accuracy or obtain a better understanding of the problems. Most of the work tried to increase accuracy by increasing the degree of freedom of the spring model (Pisano et al. 1983, Seidlitz 1989).

In 1995, Barkan and Dresner determined that the single degree of freedom model is satisfactory as long as it meets two conditions:

1. The excitation amplitudes near the first mode frequency are significantly greater than those at the second mode frequency.
2. The higher mode vibrations are not able to build up over time to high magnitudes.

Condition 1 was proven by Barkan in 1953, while condition 2 is true for most cam-follower systems where the follower rests on the cam through a large portion of the cycle or the excitations were low and internal damping was enough to damp the excitations. While many researchers have utilized the one-mass SDOF model to perform their

The advantage of utilizing a one-mass SDOF model is simplicity. However, a one-mass model does not predict the valve jump accurately. An extreme case of inaccuracy of a one-mass SDOF model was presented in 1983 by Mendez-Adriani. They found that after optimizing the system, it was possible for a one-mass SDOF model to operate at any speed without occurrences of jumps, which was not possible in practice. Therefore, a multi-mass model was created to eliminate this possibility.

### 3.1.1.2 Two-mass dynamic models

The addition of the second mass in the SDOF model allowed one to determine the contact force between the cam and follower and obtain a more accurate result. In addition, the two-mass models predict jump more accurately than the one-mass models (Barkan and Dresner 1995). While the entire mass of the one-mass model was relocated to the valve head, the masses in the two-mass model were divided and located at the valve head and the follower. The linkages’ flexibilities were modeled and included between the two masses, while the valve head spring connected the mass at the valve head to the ground. With the addition of the above parameters, a two-mass SDOF model was developed and is shown in Figure 3.3 (Barkan and Dresner 1995).

The two-mass SDOF model has the following equation of motion and contact force equation using the notation in Figure 3.3:

\[
M\ddot{x} + b\dot{x} + \left[1 + \frac{k}{K} - \frac{\dot{y}}{|\dot{y}|}\right]Kx = M\ddot{y} + k(Y + h) \tag{3.5}
\]

\[
F_c = b\dot{x} + Kx + \ddot{y}M_1 \tag{3.6}
\]

where \(x, \dot{x}, \ddot{x}\) are the displacement, velocity, and acceleration of \(M\) relative to equivalent cam follower or \(Y - y, \dot{y} - \dot{y}, \ddot{y} - \ddot{y}\), respectively.

---

**Figure 3.3 - Simplified Valve-Train**

2-Mass 1-DOF Model (Dresner & Barkan 1995)
The coulomb and viscous dampers were utilized because they improve the accuracy of the dynamic model significantly.

The benefits of two-mass model over one mass model are:

1. Allows calculation of contact force
2. Predicts jump more accurately
3. Gives a more accurate comparison to the experimental data

The disadvantages of the two-mass model were the division of the masses and more complex equations of motion, but the two-mass SDOF model was considered to be the best compromise between accuracy and complexity.

When the contact force in the two-mass SDOF model reached zero, separation occurred and the system became a two degree of freedom model or MDOF model.

### 3.1.2 Multiple degree-of-freedom model (MDOF)

A multiple degree of freedom model is a dynamic model having two or more degrees of freedom. The higher the degree of freedom, more accurate the simulation will be, at the price of a longer computation time. In order to have a realistic computation time as well as acceptable results, simplifications must be made by reducing the complexity of the model. All researchers understood this concept and tried to maximize the accuracy while minimizing the complexity and calculation time. Most of the complexities of the MDOF models of valve trains involve valve spring modeling. Because some researchers disagreed with Barkan’s decision to ignore spring surge, they included this parameter by increasing the DOF for the spring. Matsuda et al. created a one-mass model, expanded each spring into two masses, and combined them to create a five-mass MDOF model (Matsuda 1990). Seidlitz’s twenty-one degree of freedom model included nine degree of freedom for valve spring (Seidlitz 1989).

The creation of the MDOF model is similar to that of the SDOF model. Instead of combining masses, spring constants, and dampers into one lumped values, these parameters are divided and lumped into multiple values depending on the level of complexity desired. The increased in complexity allows other modes of vibrations to appear in the simulated result. According to Dresner et al., if the first mode of vibration is the dominant mode, SDOF should be used; otherwise, MDOF may be utilized. A two-
mass MDOF model, shown in Figure 3.4, was created to study the effect of system parameters by Chen et al. (1975).

![Figure 3.4 - Simplified 2-Mass 2-DOF Model (Chen et al. 1975)](image)

Figure 3.4 has the following equations of motion base on the FBD in Chen et al.:

\[ m_1 \ddot{x}_1 + (c_1 + c_2) \dot{x}_1 - c_2 \dot{x}_2 + (k_1 + k_2) x_1 - k_2 x_2 = c_1 \dot{y} + k_1 y \]  
\[ m_2 \ddot{x}_2 - c_2 \dot{x}_1 + c_2 \dot{x}_2 - k_2 x_1 + k_2 x_2 = 0 \]

(3.7)  
(3.8)

Figure 3.4 shows a two-mass two-DOF model of a form-closed system with no return spring. This model did not include a mass where the cam and follower were in contact, as seen in the work of Dresner et al., Figure 3.3. This exclusion of this mass is very similar to the one-mass SDOF model, Figure 3.2, because it can neither calculate the contact force accurately nor predict the possibility of separation. A detailed derivation of the following three-mass two DOF model was done by Norton et al. 2002:
Figure 3.5 has the following equation of motions and contact force equation based on FBD in Norton et al.:

\[
\begin{align*}
\dot{m}_1 \ddot{x}_1 + c_1 (\ddot{x}_1 - \dddot{x}_0) - c_2 (\ddot{x}_2 - \dddot{x}_1) + k_1 (x_1 - x_0) - k_2 (x_2 - x_1) &= 0 \quad (3.9) \\
\dot{m}_2 \ddot{x}_2 + c_2 (\ddot{x}_2 - \dddot{x}_1) + k_2 (x_2 - x_1) &= 0 \quad (3.10) \\
F_c = \dot{m}_0 \ddot{x}_0 + c_0 \dddot{x}_0 - c_1 (\ddot{x}_1 - \dddot{x}_0) + k_0 x_0 - k_1 (x_1 - x_0) + F_i &= 0 \quad (3.11)
\end{align*}
\]

MDOF models allow more accurate data to be obtained when compared to the SDOF model. Data such as stresses and loads in the linkages are obtained with an MDOF. The MDOF model should be utilized when the first mode of vibration is not predominant or a higher degree of accuracy is required. The benefits of an MDOF model over an SDOF model are:

1. Predict the dynamic behavior more accurately
2. Allow calculations of stresses between linkages
3. Maintain the benefits of SDOF model with the exception of modeling complexity or simulation time needed.
3.2 Impact Modeling

Impact modeling is the modeling of impact force that occurs due to a collision of a rigid and elastic, or of two elastic bodies. The first time that impact was considered in the valve-train model was in 1953 by Barkan. Although Barkan knew that the valve-head would impact the valve seat, he ignored the impact events and assumed the first impact event as the end of the analysis. He deemed the impact due to valve seating as irrelevant to the motion of the valve head during the period of interest. The valve-seat condition was also excluded by Matsuda et al. (1990) in their analysis of the five mass MDOF model. In 1995, Dresner and Barkan also neglected the seat flexibility from their analysis because they felt that the seat deflection contributes only during lift off and seating. They considered valve seating as the end of their analysis, which was a justification for the exclusion. Even though they excluded the valve seating impact, they included the cam jump with a coefficient of restitution of zero. This allowed researchers to determine the jump conditions but not the effects of impact, which may be appropriate depending on the motivation of the analysis.

Pisano and Freudenstein, 1983, included valve-seat flexibility in their model but their most important contribution was the distributed parameters of the valve spring, which gave a more accurate result. The impact modeling technique shown in the published literature used the Heaviside step function, which was easily implemented and gave accurate results. The valve-seat was represented by a stiffness constant and viscous damping in the dynamic model. The calculations of how these values were obtained were also excluded from the literature, which made it hard to follow. Seidlitz continued the inclusion of the valve-seating as well as increasing the complexity of the valve spring. This was an attempt to obtain an optimal model that could be used to calculate the dynamics of the cam-follower system for valve-train at various speeds in 1989. These two methods may lack extensive derivation but the approaches were ones which can easily be found in other literature. The same could not be said for Bagepalli et al. In 1991, Bagepalli et al. used Uicker’s method of relative coordinates to model the cam-follower pairs in mechanisms. Based on other approaches seen previously, this method was the most complex. Not only was Uicker’s method used, but Bagepalli et al. also used a Hertzian spring and damper to represent the effect of impact. The Hertzian
contact spring is a non-linear spring, which further complicates the already complex problem.

Although Pisano et al., 1983, and Seidlitz, 1989, included the effects of impact on the system, neither of them included or mentioned the determination of the impact force. Luckily, the impact force calculations as well as the effects were extensively researched in the field of industrial manipulators. In 1987, Wang and Mason studied impact forces, the effect of object orientation, and direction of impact (Wang and Mason 1987). The study was performed for unconstrained planar collisions that included various possible outcomes as a result of the impact which were sliding contact, sticking contact, and reversed sliding contact. Unfortunately, the research did not include a comparison between the simulated impact force and the experimental data. However, this literature may be very useful for a higher degree of freedom impact pairs which may be considered for future research. The energy method was used by Youcef-Toumi and Gutz to calculate the impact force (Youcef-Toumi and Gutz 1994). For the determination of the impact force, a coefficient of restitution was used following Goldsmith’s book on impact. The simulated model was transformed into a dimensionless form. The impact force was also modeled as a spring and damper which influenced the motion during the contact. The correlation was not as accurate as that obtained by some other research, and the dimensionless approach may be appropriate in a few situations but not necessarily in the modeling of impact. The most extensive impact force determination was completed by Ferretti et al. in Impact Modeling and Control for Industrial Manipulators (Ferretti et al. 1998). The experimental data compared to the simulated model was force data obtained through an input from industrial manipulator motor impacting a hard granite surface. The links were assumed rigid while the joints were modeled as springs. The impact modeling techniques were very similar to the one used in cam-follower mechanism literature. Spring and damper elements were modeled in place of the force sensor used in the experiment. The comparisons were satisfactory even though the dynamic effects were not shown in the literature. The research on the modeling of impact force for industrial manipulators were more concerned with the control of the vibration and instability, which may have occurred due to the impact which inadvertently redirected the concentration from the periodic effects such as those seen in cam-follower systems. This could be
considered as the explanation of the exclusion of the dynamic effects because the intention was not to allow vibration to continue but to control it immediately.

It appeared that the actual impact force determination was not one of the criteria in the cam-follower models, whereas the dynamic effects of the impact were not critical in the calculation in industrial manipulators. By employing the impact force approximation of industrial manipulators literature and the dynamic effects of the impact of cam-follower literatures, it was possible to create a dynamic model of a cam-follower system with impact loading. Although, only the energy method was shown in the calculation of impact force in the industrial manipulator, other methods are also available and described in the following sections.

### 3.3 Impact Solvers

Impact force is a force that occurs due to a rapid application of load over a small time interval where the stress travels away from the location of impact (Goldsmith 1960). There are two main types of impacts, a striking impact and a force impact. A striking impact occurs when two bodies, which are not in contact, engage in a collision, whereas a force impact occurs when both bodies are in contact with no force and the support is suddenly removed causing a sudden application of force at zero relative velocity. By comparing the calculated impact force to the yield strength of the struck object, it is possible to determine the life expectancy of the object and a possibility of redesigning the object with a more appropriate material. Energy and Wave analyses are the two main methods in solving impact forces and are discussed in the following sections.
3.3.1 Energy methods for impact modeling

![Diagram of force vs. deflection curve]

Figure 3.6 – Force vs. Deflection (Burr 1982, p. 591)

The energy method states that the sum of the kinetic energy of the colliding masses prior to impact equals the sum of the potential energy of the collided masses. To understand the problem better, the impact was divided into four stages. The initial stage is when the masses are not in contact but are moving toward each other. The second stage occurs when the parts are in contact but no deformation has occurred; this is where the impact velocity is obtained. Once the masses reach the maximum deformation, all the energy has been stored in a form of potential energy and the masses are moving at the same velocity. All of the potential energy is released and converted into kinetic energy by pushing the two masses away from each other. The stored energy can be determined by calculating the area under the force vs. deflection curve.

In the case of a linear elastic force versus deflection curve, Figure 3.6, the potential energy is simply:

\[ P.E. = \int_{0}^{\delta_{\text{max}}} F_{\text{max}} dy = \frac{F_{\text{max}} \delta_{\text{max}}}{2} \]  \hspace{1cm} (3.12)

where \( F_{\text{max}} \) is the maximum impact force and \( \delta_{\text{max}} \) is the maximum deflection.

Since the relationship of force versus deflection is known:
\[ F = k \times \delta \quad \text{or} \quad \delta = \frac{F}{k} \quad (3.13) \]

It is possible to obtain a simpler potential energy equation:

\[ P.E. = \frac{F_{\text{max}}^2}{2k} \quad (3.14) \]

where \( k \) is the elasticity of the impacted system.

There are multiple approaches to using an energy method. The simplest one would be the common velocity approach, which gives the range of maximum impact force. Unfortunately, this method requires further analysis to determine the maximum impact force (Johnson 1958). A more complex approach requires deflection determination through a mechanical analysis or a finite element analysis program and a correction factor. The final result of this method gives an estimated maximum impact force. This value may be relatively close to the actual impact force, but it certainly does not match it (Burr 1982). The most complex method is probably the common velocity method with the additions of energy loss due to structural damping and strain energy (Kahng & Amirouche 1987). In most cases, it is assumed that the energy loss due to structural damping is insignificant compared to the impact force, therefore this method will not be discussed here. Any of these methods maybe utilized depending on modeling complexity desired by the designer.

### 3.3.2 Deflection and correction factor approach for impact modeling

The deflection approach with the correction factor has the governing equation for a simple vertical impact system, shown in Figure 3.7, of:

\[ \frac{F_{\text{max}}^2}{2k} = \frac{\eta m_1 v_0^2}{2} + W' \delta_{\text{max}} \quad (3.15) \]

where \( \eta \) is the correction factor, \( m_1 \) is the mass of the driving object, \( v_0 \) is the velocity at the point of impact, and \( W' \) is the weight of

![Figure 3.7 - Striking Impact – Vertical Fall (Burr 1982, pp. 591)](image)
the moving mass. This equation takes into account the static deflection that occurred due to the weight of moving mass as well as energy loss in various mass ratios through the implementation of $\eta$. When the governing equation was simplified and substituted with the appropriate conditions, the impact force equation was:

$$\frac{F_{\text{max}}}{W'} = 1 + \sqrt{1 + \frac{2\eta h}{\delta_{st}}}$$

(3.16)

where $h$ is the height at which the mass was dropped and $\delta_{st}$ is the static deflection. The above equation shows that when two masses are in contact and one of the mass’s support is removed instantly, $h = 0$, the impact force is two times the weight.

The correction factor $\eta$, for a system shown in Equation 3.17, is calculated through the kinetic energy balance right after impact. The assumption was made that the elastic member’s velocity particles are proportional to their deflection under a static load applied at the location of impact. This assumption essentially states that:

$$u = v_a \left(\frac{y}{l}\right)$$

(3.17)

where $u$ is the velocity of the particle and $v_a$ is the velocity of the mass right after the impact. The kinetic energy of this system can be described by:

$$E_{\text{kinetic}} = \frac{m_1 v_a^2}{2} + \int_0^l \rho A dy \left(\frac{v_a}{l}\right)^2 = \frac{v_a^2}{2} \left(m_1 + \frac{m_2}{3}\right)$$

(3.18)

where $\rho$ is the mass density and $m_2$ is the mass of the bar. By incorporating the conservation of momentum, $m_1 v_0 = (m_1 + m_2 / 3) v_a$, it was possible to obtain the following equation:

$$E_{\text{kinetic}} = \frac{m_1 v_0^2}{2} \left(\frac{m_1}{m_1 + m_2 / 3}\right)$$

(3.19)

By comparing the above equation to Equation 3.15 kinetic energy component, it can be observed that the correction factor is:
\[ \eta = \left( \frac{m_1}{m_1 + m_2 / 3} \right) \] (3.20)

The impact force equation of this method could be simplified to:

\[ F_{\text{max}} = \left( 1 + \sqrt{1 + \frac{2m_1 h}{(m_1 + m_2 / 3)\delta_{st}}} \right)W' \] (3.21)

The above equation is applicable to a detached mass dropped from a specific height. In order to obtain an equation that is utilizable in cases of known impact velocity, the following substitution should be considered:

\[ v_0^2 = v_i^2 + 2gh \quad \text{when } v_i = 0 \quad h = \frac{v_0^2}{2g} \] (3.22)

by making appropriate substitutions it is possible to obtain an impact force equation that is not dependent upon the height, \( h \).

\[ F_{\text{max}} = \left( 1 + \sqrt{1 + \frac{m_1v_0^2}{(m_1 + m_2 / 3)g\delta_{st}}} \right)W' \] (3.23)

The most complex part of this method would be determination of the static deflection of the non-moving mass. For a simple problem shown above, it could easily be determined by calculating the linear deformation shown in Equation 3.32 and 3.33. However, for a more complex system such as an over-travel mechanism, it would be impossible to manually calculate the deformation unless the problem is overly simplified, which would result in an inaccurate value. If static deformation could be calculated easily, this method would be superior because it contains very few variables and only one unknown variable. This method may be relatively accurate but it is not exact. Therefore, it may be appropriate to obtain the range of the impact force that a system is capable of exerting. This range of impact force could be obtained through the use of the common velocity method.
3.3.3 The common velocity approach for impact modeling

In most cases where it is impossible to calculate the exact value for the impact force, it may be desirable to determine the range of the impact force that is applied to the system (Johnson 1958). This approach is superior when the exact values of several variables cannot be determined easily and it is necessary to obtain the range of impact force immediately. Finite and infinite mass assumptions assume a finite value or an infinite value for driving mass, respectively. Calculating the finite mass equation will give a lower impact force boundary while calculating the infinite mass equation will give an upper impact force boundary. It is known that when the maximum deformation is obtained, their relative velocity is zero and their common velocity is:

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v_c \Rightarrow v_c = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} \quad (3.24)$$

where \(m_1, m_2, v_1, v_2,\) and \(v_c\) are the driving mass, driven mass, driving mass velocity, driven mass velocity, and common velocity, respectively. The energy stored is the difference between the sum of the kinetic energy of the masses at the instant of impact and the kinetic energy at the instant of maximum deformation:

$$E_{stored \_lower} = \frac{1}{2} \left[ (m_1v_1^2 + m_2v_2^2) - (m_1 + m_2)v_c^2 \right] \quad (3.25)$$

The above energy is stored in the elastic region, which equals:

$$\frac{F_{max \_lower}^2}{2k} = E_{stored \_lower} = \frac{1}{2} \left[ (m_1v_1^2 + m_2v_2^2) - (m_1 + m_2)v_c^2 \right] \quad (3.26)$$

Through manipulation, it is possible to determine that the lower boundary of the impact force with finite-mass assumption:

$$F_{max \_lower} = \sqrt{k \left[ (m_1v_1^2 + m_2v_2^2) - (m_1 + m_2)v_c^2 \right]} \quad (3.27)$$

In the case of infinite mass, at the maximum deflection, there would be no change in velocity due to impact. Since the driving mass is infinite, \(m_1 = \infty,\) the only velocity change observed would be in the driven mass, and therefore the change in kinetic energy would be:

$$E_{stored \_upper} = \frac{m_2}{2} (v_1 - v_2)^2 \quad (3.28)$$
By setting equation 3.28 equal to the energy stored in the elastic region, it is possible to determine the upper limit of the maximum impact force:

\[
\frac{F_{\text{max, upper}}^2}{2k} = \frac{m_2}{2}(v_1 - v_2)^2 \quad \Rightarrow \quad F_{\text{max, upper}} = \left(v_1 - v_2\right)\sqrt{km_2}
\] (3.29)

Once the upper and lower ranges of the impact force are obtained, it is possible for the designer to utilize this information to optimize the design to account for the maximum impact force possible.

The two methods shown above are valid only when the deformation of the elastic region is linear. Unfortunately, obtaining linear deformation is nearly impossible. The problems of trying to obtain a plane-to-plane contact linear deformation are manufacturing tolerances, surface finish, and assembly errors. Therefore, a non-linear deformation should be utilized instead. For a spherical surface member and a linear deformation member contacting a flat surface, the impact force equation is:

\[
F = C\delta^a
\] (3.30)

where \(F\), \(C\), \(\delta\), and \(a\) are the impact force, constant of proportionality, combined deformation, and the slope of the force versus deformation curve plotted on a log-log scale, respectively.

The total deformation is the sum of the linear and non-linear deformation:

\[
\delta = \delta_{\text{linear}} + \delta_{\text{sphere}}
\] (3.31)

Linear deformation of an axially loaded member and the non-linear deformation of a spherical surface can be calculated, respectively, by the following equations:

\[
\delta_{\text{linear}} = \frac{FL}{AE}
\] (3.32)

\[
\delta_{\text{sphere}} = 1.55 \left(\frac{F^2}{E^2D}\right)^{\frac{1}{3}}
\] (3.33)
where \(L, A, E,\) and \(D\) are the length of the overhanging linear member, the cross-sectional area of the linear member, modulus of elasticity, and diameter of the sphere and the linear member, respectively.

Once the force versus deflection data is obtained on a log-log scale, a plot can be made and the slope, \(a\), obtained. The constant of proportionality can be calculated by substituting \(a\) into Equation 3.30. Through substitution of Equation 3.12, 3.26, and 3.30, it is possible to obtain the lower limit of the impact force through a finite-mass assumption:

\[
F_{\text{max,lower}} = C(V_{\alpha+1}) \left\{ \frac{a+1}{2} \left[ (m_1v_1^2 + m_2v_2^2) - (m_1 + m_2)v_c^2 \right] \right\}^{(\psi_{\alpha+1})}
\]

(3.34)

while the substitution of Equation 3.13, 3.28, and 3.30 give the upper limit of the impact force through infinite-mass assumption:

\[
F_{\text{max,upper}} = C(V_{\alpha+1}) \left\{ \frac{a+1}{2} m_2 (v_1 - v_2)^2 \right\}^{(\psi_{\alpha+1})}
\]

(3.35)

Through the use of the above equations, it is possible to determine the range of the impact force for a non-linear system. Although, this method may be more complex to calculate due to multiple variables, it is preferred in most situations because it eliminates unaccounted errors by creating an approximate point contact instead of planar contact.

As seen in the above impact force determination, energy methods are fairly simple to apply and are not time dependent. The advantages of using an energy method are:

1. It assumes that maximum stress occurs simultaneously throughout the elastic member which simplifies the problem greatly.

2. It does not include the rapid surging because it assumes a high ratio of rigid mass to elastic region mass.

3. It neglects the retarding forces such as friction assuming its magnitude is insignificant compared to the impact force.

These assumptions allow the designer to obtain the approximate impact force without the complexities of the wave method. If the most precise approximation is needed and time is not a constraint, the wave method should be utilized. The wave method is described in the next section.
3.3.4 The wave method for impact modeling

The wave propagation method is a more accurate and complex method than the energy method (Burr 1982). This method is time dependent because it calculates the stress waves that travel through the elastic media. The velocity at which the stress travels depends on the material property:

\[ v_s = \sqrt{\frac{E}{\rho}} \]  

(3.36)

where \( v_s \) is the stress velocity, \( E \) is the modulus of elasticity, and \( \rho \) is the mass density. The above velocity is taken as the velocity of sound in this particular medium and the waves move together through parallel planes. The stress and velocity travel in two directions, positive and negative as shown in Figure 3.10.

![Wave Method Notations](image)

where \( A \) is the cross-section area, \( u \) is the particle displacement, \( \frac{\partial u}{\partial x} \) is strain, \( \frac{\partial u}{\partial t} \) is particle velocity, and \( \frac{\partial^2 u}{\partial t^2} \) is particle acceleration.

Balancing the above diagram gives:

\[ AE \left[ \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) dx \right] - AE \frac{\partial u}{\partial x} = Adx \rho \frac{\partial^2 u}{\partial t^2} \]  

(3.37)

Manipulating the equation of motion gives a second order differential equation of:

\[ \frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u}{\partial x^2} = v_s^2 \frac{\partial^2 u}{\partial x^2} \]  

(3.38)

The above equation is a one-dimensional wave equation, which is linear and homogeneous. Its solution has the following form:

\[ u(x, t) = f(x - ct) + f'(x + ct) \]  

(3.39)
where $f$ is the positive velocity and $f'$ is the negative velocity. The solution is the sum of positive and negative waves which travel to the right and left, respectively.

The equation was further simplified to solve for the velocity and stress relationship. For a positive wave, $f$, stress is $s$ and velocity is $v$. Through the manipulation of the known equations, it was possible to obtain the following stress-velocity relation:

$$v = -\frac{s}{\sqrt{E \rho}} \quad \text{or} \quad s = -v\sqrt{E \rho} \quad (3.40)$$

The same process was applied to the negative wave, $f'$, where stress is $s'$ and velocity is $v'$. Through the manipulation of the known equations, it was possible to obtain the following stress-velocity relation:

$$v' = \frac{s'}{\sqrt{E \rho}} \quad \text{or} \quad s' = v'\sqrt{E \rho} \quad (3.41)$$

The net stress is determinable at any point in time by summing the positive and negative stresses:

$$\sigma = E \frac{\partial u}{\partial x} = E \left( \frac{\partial f}{\partial x} + \frac{\partial f'}{\partial x} \right) = s + s' \quad (3.42)$$
Figure 3.11 shows the driving mass, \( m_1 \), approaching the driven mass. The initial stress at the instant of impact acts in the opposite direction as the velocity. It is possible to obtain the initial stress from:

\[
\sigma_0 = v_0 \sqrt{E\rho} \quad @ \quad x=l
\]  

(3.43)

because there would be no positive velocity at the instant of impact or \( s = 0 \). A better understanding may be obtained by summing the forces acting in Figure 3.11 at the instant of impact.

\[-As' = m_1 a = m_1 \frac{dv'}{dt} = m_1 \frac{ds'}{\sqrt{E\rho}dt} \quad \frac{ds'}{dt} + \frac{A\sqrt{E\rho}}{m_1} s' = 0\]

(3.44)

Simplification of the above equation can be made by substituting \( q' \) in place of \( \frac{A\sqrt{E\rho}}{m_1} \):

\[\frac{ds'}{dt} + q's' = 0\]

(3.45)

This first order differential equation has a solution in the form of:

\[s' = C_1 e^{-q't}\]

(3.46)

\( C_1 \) is the constant of integration which can be evaluated by substituting the initial condition \( s' = \sigma_0 \). There is no positive wave present at the instant of impact. Therefore, the final solution for the stress in this case consists only of the negative component and is:

\[\sigma_{x=l} = s' = \sigma_0 e^{-q't}\].

Once the stress is obtained, the velocity can easily be derived through the use of Equation 3.41:

\[v'_{x=l} = \frac{\sigma_0}{\sqrt{E\rho}} e^{-q't}\]

(3.47)

It has already been established that the velocity of the wave travel is \( C = \sqrt{E / \rho} \). The period, \( T \), that the wave will travel to the end of an elastic member, \( x=0 \), with a length of \( l \) and back is simply \( 2 \times \frac{l}{C} \). The next step of calculation depends on the condition at \( x=0 \).
<table>
<thead>
<tr>
<th>Condition @ x = 0</th>
<th>Constraint</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free</td>
<td>( \sigma_0 = 0 )</td>
<td>( s = -s' = -\sigma_0 e^{-\theta(t-0.5T)} )</td>
</tr>
<tr>
<td>Fixed to Infinite Mass</td>
<td>( V = 0 )</td>
<td>( s = s' = \sigma_0 e^{-\theta(t-0.5T)} )</td>
</tr>
<tr>
<td>Attached to Finite mass</td>
<td>( V = \frac{1}{\sqrt{E\rho}} \left(-s + s'\right) )</td>
<td>( s = s' - V \sqrt{E\rho} )</td>
</tr>
</tbody>
</table>

Table 1 - Boundary Conditions for Wave Method

The boundary condition for \( x = 0 \), fixed to infinite mass, was further explored because this was the boundary condition for the impact mechanism. To simplify this complex equation, multiple substitutions were applied. It was determined that

\[
s' = \sigma_0 e^{-(2\alpha/\lambda)\zeta}
\]

where \( \alpha \) is the ratio of elastic mass to driven mass, \( \alpha = \frac{m_h}{m_2} \), \( \lambda \) is the ratio of driving mass to driven mass, \( \lambda = \frac{m_1}{m_2} \), and \( \zeta \) is the ratio of time of interest to the wave travel period, \( \zeta = t/T \). The force at the instant of impact is what most designers are interested in. Therefore, it was shown that the force at the instant of impact when the driven mass is infinite is:

\[
F_i = \sigma_o A \left( 1 + \frac{1}{\sqrt{\beta}} + \frac{2}{3} \right)
\]

where \( \beta \) is the ratio of elastic mass to the driving mass, \( \beta = \frac{m_h}{m_1} \). The wave propagation impact force equation was proven to have a higher impact force than the deflection and correction factor approach of the energy method. The higher impact force predicted may help prevent a failure of a mechanism because other methods underestimate the actual impact force whereas the wave method is more accurate.
4 TEST APPARATUS

4.1 Existing CDTM

The machine used for experimentation is the Cam Dynamics Test Machine (CDTM). It was designed, manufactured, and assembled as a part of M.A. Munyon’s directed research project. The output data were divided into three groups; shaft data, primary cam, and secondary cam. The output signals of the shaft data are driveshaft position, torque, and camshaft position, while the output signals for primary and secondary cams are positions, velocities, and accelerations as depicted in Figure 4.1.

![Figure 4.1 - Overview of the Original CDTM](image-url)
Since this thesis project is required to have similar characteristics to an assembly machine, which consists of impact and over-travel, minor modifications were made to the CDTM to include impact and over-travel. The CDTM is equipped with a speed-controlled DC 1750 RPM, 1 horsepower motor, which provides a relatively constant rotational speed through the adjustable speed controller. The ratio of the motor flywheel to the flywheel that drives the cam-shaft is 7:20. This rotational output provides 120 RPM to the camshaft for the experiments. The current cam profile is divided into eight segments, which are:

<table>
<thead>
<tr>
<th>Segment #</th>
<th>Motion</th>
<th>Function</th>
<th>Beta (Deg)</th>
<th>Linear (in)</th>
<th>Angular (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Rise</td>
<td>4567 Poly</td>
<td>50</td>
<td>0.50</td>
<td>4.40</td>
</tr>
<tr>
<td>2</td>
<td>Dwell</td>
<td>-</td>
<td>40</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>3</td>
<td>Fall</td>
<td>345 Poly</td>
<td>50</td>
<td>-0.50</td>
<td>-4.40</td>
</tr>
<tr>
<td>4</td>
<td>Dwell</td>
<td>-</td>
<td>40</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>5</td>
<td>Rise</td>
<td>Mod Trap</td>
<td>50</td>
<td>0.50</td>
<td>4.40</td>
</tr>
<tr>
<td>6</td>
<td>Dwell</td>
<td>-</td>
<td>40</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>7</td>
<td>Fall</td>
<td>Mod Sine</td>
<td>50</td>
<td>-0.50</td>
<td>-4.40</td>
</tr>
<tr>
<td>8</td>
<td>Dwell</td>
<td>-</td>
<td>40</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 2 - CDTM Motions, Functions, and Characteristics

A visual representation of the cam is needed to better understand the function of the input of the CDTM. Therefore, the displacement, velocity, and acceleration versus time were plotted for CDTM operating at 120 RPM and shown in Figure 4.2 through Figure 4.4.
Figure 4.2 - Input Function: Displacement vs. Cam Angle

Figure 4.3 - Input Function: Velocity vs. Cam Angle

Figure 4.4 - Input Function: Acceleration vs. Cam Angle
The prime radius of the cam is approximately 3.5 inches. Both cam-followers systems are attached with Macro Sensors linear variable differential transformer (LVDT) model number DC750-500\(^1\), Trans-Tek linear velocity transformer (LVT) model number 0112-0000\(^2\), and a DYTRAN piezoelectric accelerometer model number 3145A\(^3\). A DYTRAN piezoelectric accelerometer model number 3035A was also used in the situations where the impact and over-travel events were included in the experiment because impact events overload the 3145A\(^4\). The LVDT and LVT are attached by the mean of pins at a distance of 9 inches and 10 inches, respectively, from the ground pivot on the link arm. The piezoelectric accelerometers were attached by the means of a ‘stud’ on the rocker arm, which gave it a maximum of 7 kHz frequency bandwidth and was at a distance of 9.00 inches from the connection of the small rod end.

![Figure 4.5 - Original Dimensioned CDTM with Sensors](image)

---

4.2 Redesigned CDTM with Impact and Over-travel

An impact and over-travel mechanism was designed, manufactured, and assembled by the author on the CDTM. The most important requirement was to ensure that the impact and over-travel mechanism was implemented and the events were consistent and repeatable. The design must not exceed the available space or interfere with the existing machine as well as minimizing the modification made to the existing parts. Adjustability of the over-travel distance is highly desirable because this will allow the user to change the over-travel distance, impact velocity, and force exerted on the hard-stop. The over-travel spring must prevent the separation of the impact mass and the intermediate mass until impact occurs to ensure that the condition experienced in the actual system is maintained in the experiment. Lastly, the force experienced by the impact mechanism must be obtainable through experiment, which is done by placing the force sensor where the striking impact occurs.

When the requirements were considered and multiple initials designs were created, the final design that was manufactured and assembled is shown in Figure 4.6 and Figure 4.7.

![Figure 4.6 - Isometric View of the Impact and Over-Travel Mechanism](image)

1. Extended base
2. Stanchion
3. Stanchion rib
4. THK rail
5. Hard-stop
6. Force transducer
7. Impact mechanism
8. Over-travel mechanism
Figure 4.7 - Overview of the Impact and Over-Travel Mechanism

Parts number one through six are non-moving parts, while seven and eight are moving sub-assemblies. The extended base was designed so that it could be mounted onto the existing top plate with the only alteration being the use of longer socket cap screws that
connect the extended base and the existing top plate to the side plates. The stanchion was designed to be parallel with the motion of the intermediate mass and is mounted on the extended base, while the stanchion rib provides a counter moment to the impact. The THK rail was a standard part which was implemented to ensure that the impact motion is a linear one and induces minimal resistance. The hard-stop was created to provide sufficient stiffness to minimize deformation due to impact and allow the mounting of the force transducer. The force transducer was also a standard part that was chosen after the approximate impact force was determined.

![Impact Mechanism Components](image)

**Figure 4.8 - Impact Mechanism Components**

The THK cart rides on the THK rail which provides an almost frictionless motion, while the rod end block was designed to mount onto the THK rail. The hex nut and the impact screw can be adjusted and allow multiple over-travel distance. The most complex mechanism to design was the over-travel mechanism shown in Figure 4.9 and Figure 4.10.
Figure 4.9 shows an exploded view of the over-travel mechanism. Top rod end (8.1), adaptor (8.2), and enclosure sleeve (8.6) are connected as one sub-assembly while the bottom rod end (8.7), shoulder screw (8.3), and washer (8.4) are considered as another sub-assembly. A die spring with stiffness constant of 225 lb/in was chosen after the maximum non-impact force was determined. This was calculated by obtaining the maximum linear acceleration at the axis of impact and multiplying by the impact mass.

The extended base, stanchion, stanchion rib, hard stop, and rod end block are made of aluminum, while the adaptor and enclosure sleeve are made of hexagonal brass. Their exact dimensions are in Appendix F. The THK rail and cart model number is HSR
10RM. The rail allows the cart to travel linearly at a maximum distance of approximately 5 inches.

Figure 4.10 – Sectioned View: Over-Travel Mechanism and Force vs. Deflection Plots

Figure 4.10 shows a sectioned view of the over-travel mechanism. Because the two sub-assemblies, top rod end and bottom rod end sub-assemblies, are not rigidly connected but are in contact under a preload force from die spring; this introduces two stiffness constants of the over-travel mechanism, pushing and pulling. The rising motion of the cam results in a force from the bottom rod end being exerted on the enclosure sleeve, scenario 1 of Figure 4.10. During the fall motion of the cam, the washer is exerting force on the spring which in turn exerts the same force on the enclosure sleeve to pull the THK cart down, scenario 2 of Figure 4.10.

In order to determine the stiffness constants of over-travel mechanism, lumped stiffness constants of the pushing and pulling motion must be calculated. The pushing motion requires the stiffness constant of the screw attaching the bottom rod end to the left side of the arm rocker, bottom rod end, enclosure sleeve, adaptor, top rod end and the screw attaching the rod end to the impact block. The pulling motion consists of the deflection of the bottom screw, bottom rod end, shoulder screw and washer, spring,

---

enclosure sleeve, adaptor, top rod end, and the top screw. These calculations are shown in Appendix E. The force transducer fastened to the hard stop is a Dytran 1050V3 LIVM sensor with an operating range of ± 100 lbf\(^6\). Since the actual sensitivity may vary from the theoretical sensitivity, Table 3 represents the sensors being used in the CDTM, their model numbers, ranges, and actual sensitivities.

![Figure 4.11 - Final Dimensioned CDTM with Sensors (Parts Hidden)](image)

<table>
<thead>
<tr>
<th>Name</th>
<th>Measure</th>
<th>Model Number</th>
<th>Range</th>
<th>Sensitivity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Macro Sensors</td>
<td>Displacement</td>
<td>DC750-500</td>
<td>±0.5 in</td>
<td>21.38 V/in</td>
</tr>
<tr>
<td>Trans-Tek</td>
<td>Velocity</td>
<td>0112-0000</td>
<td>2.0 in</td>
<td>550 mV/ in/s</td>
</tr>
<tr>
<td>Dytran</td>
<td>Acceleration</td>
<td>3145A</td>
<td>50g</td>
<td>101 V/g</td>
</tr>
<tr>
<td>Dytran</td>
<td>Acceleration</td>
<td>3035A</td>
<td>500g</td>
<td>10.3 V/g</td>
</tr>
<tr>
<td>Dytran</td>
<td>Force</td>
<td>1050V3</td>
<td>± 100 lbf</td>
<td>53.9 V/lbf</td>
</tr>
</tbody>
</table>

Table 3 - CDTM Sensors' Ranges and Sensitivities

---

### Variables Used in Chapter 5 onward:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{la}$</td>
<td>Mass of Link Arm</td>
</tr>
<tr>
<td>$m_{cr}$</td>
<td>Mass of Connecting Rod</td>
</tr>
<tr>
<td>$m_{arr}$</td>
<td>Mass of Arm Rocker (Right side)</td>
</tr>
<tr>
<td>$m_{arl}$</td>
<td>Mass of Arm Rocker (Left side)</td>
</tr>
<tr>
<td>$m_{bre}$</td>
<td>Mass of Bottom Rod End</td>
</tr>
<tr>
<td>$m_{tre}$</td>
<td>Mass of Top Rod End</td>
</tr>
<tr>
<td>$m_1$</td>
<td>Sum of Link Arm and Connecting Rod Masses</td>
</tr>
<tr>
<td>$m_2$</td>
<td>Mass of Arm Rocker (Right side)</td>
</tr>
<tr>
<td>$m_3$</td>
<td>Sum of Arm Rocker (Left side) and Bottom Rod End Masses</td>
</tr>
<tr>
<td>$m_4$</td>
<td>Mass of Top Rod End</td>
</tr>
<tr>
<td>$M_1$</td>
<td>Follower Mass</td>
</tr>
<tr>
<td>$M_2$</td>
<td>Intermediate Mass</td>
</tr>
<tr>
<td>$M_3$</td>
<td>Impact Mass</td>
</tr>
<tr>
<td>$k_{cs}$</td>
<td>Stiffness of Closure Spring</td>
</tr>
<tr>
<td>$k_{cr}$</td>
<td>Stiffness of Connecting Rod</td>
</tr>
<tr>
<td>$k_{arr}$</td>
<td>Stiffness of Arm Rocker (Right side)</td>
</tr>
<tr>
<td>$k_{arl}$</td>
<td>Stiffness of Arm Rocker (Left side)</td>
</tr>
<tr>
<td>$k_{bre}$</td>
<td>Stiffness of Bottom Rod End</td>
</tr>
<tr>
<td>$K_{01}$</td>
<td>Stiffness between the Ground and Follower Mass</td>
</tr>
<tr>
<td>$K_{03}$</td>
<td>Stiffness between the Ground and Impact Mass</td>
</tr>
<tr>
<td>$K_{12}$</td>
<td>Stiffness between the Follower Mass and Intermediate Mass</td>
</tr>
<tr>
<td>$K_{23}$</td>
<td>Stiffness between the Intermediate Mass and Impact Mass</td>
</tr>
<tr>
<td>$c_{cs}$</td>
<td>Damping of Closure Spring</td>
</tr>
<tr>
<td>$c_{cr}$</td>
<td>Damping of Connecting Rod</td>
</tr>
<tr>
<td>$c_{arr}$</td>
<td>Damping of Arm Rocker (Right side)</td>
</tr>
<tr>
<td>$c_{arl}$</td>
<td>Damping of Arm Rocker (Left side)</td>
</tr>
<tr>
<td>$c_{bre}$</td>
<td>Damping of Bottom Rod End</td>
</tr>
<tr>
<td>$c_{01}$</td>
<td>Damping between the Ground and Follower Mass</td>
</tr>
<tr>
<td>$c_{12}$</td>
<td>Damping between the Follower Mass and Intermediate Mass</td>
</tr>
<tr>
<td>$c_{23}$</td>
<td>Damping between the Intermediate Mass and Impact Mass</td>
</tr>
<tr>
<td>$s$</td>
<td>Input Displacement</td>
</tr>
<tr>
<td>$x_1$</td>
<td>Displacement of Follower Mass ($M_1$)</td>
</tr>
<tr>
<td>$x_2$</td>
<td>Displacement of Intermediate Mass ($M_2$)</td>
</tr>
<tr>
<td>$x_3$</td>
<td>Displacement of Impact Mass ($M_3$)</td>
</tr>
<tr>
<td>$OTD$</td>
<td>Over-travel Distance</td>
</tr>
<tr>
<td>Preload</td>
<td>$K_{23}$ (Die Spring) Preload</td>
</tr>
<tr>
<td>$dis_{imp}$</td>
<td>Simulated Displacement with Impact</td>
</tr>
<tr>
<td>$dis_{no_imp}$</td>
<td>Simulated Displacement without Impact</td>
</tr>
<tr>
<td>$dis_{imp-max}$</td>
<td>Maximum Simulated Displacement with Impact</td>
</tr>
<tr>
<td>$vel_{imp}$</td>
<td>Simulated Velocity with Impact</td>
</tr>
<tr>
<td>$vel_{no_imp}$</td>
<td>Simulated Velocity without Impact</td>
</tr>
<tr>
<td>$vel_{imp-max}$</td>
<td>Maximum Simulated Velocity with Impact</td>
</tr>
<tr>
<td>$acc_{imp}$</td>
<td>Simulated Acceleration with Impact</td>
</tr>
<tr>
<td>$acc_{no_imp}$</td>
<td>Simulated Acceleration with Impact</td>
</tr>
<tr>
<td>$acc_{imp-max}$</td>
<td>Maximum Simulated Acceleration with Impact</td>
</tr>
</tbody>
</table>
5 MODELING OF CDTM

Once the best impact force approximation was found (Appendix C) thorough derivations of the new dynamic model was performed. A few assumptions were made to simplify the problem; these assumptions were that mass $M_1$ was always in contact with the cam, the preload force of the spring $K_{23}$ maintained contact between the impact mass $M_3$ and the intermediate mass $M_2$ up to the point of impact, and the condition of impact did not change from wear caused by multiple impacts. With these assumptions, detailed derivations of the dynamic model were performed for three possible conditions and are presented and discussed in this section.

5.1 Universal Schematic and Free Body Diagram

Prior to dividing the problem into multiple segments, it was important to obtain an overview of the problem by identifying the forces acting on each mass under every circumstance.

Figure 5.1 - Universal Diagram of CDTM

Figure 5.1 shows a schematic diagram and free body diagram (FBD) of CDTM with every force identified. However, it is impossible to have every force in Figure 5.1
acting on the system simultaneously. Therefore, depending on the conditions, which are discussed in the following sections, certain forces may not be acting on CDTM. After every condition is clearly understood, universal equations of motion will be developed.

5.2 No Contact

Figure 5.2 - Diagram of CDTM: Condition 1

Figure 5.2 shows the FBD of the CDTM in the first condition which is valid only when the impact mass $M_3$ is not in contact with the seat, $x_2 = x_3 > OTD$, where $OTD$ is a fixed predefined over-travel distance. The forces acting on the system in this condition can be divided into three categories; contact force ($F_c$), spring force ($F_{kij}$), and damping force ($F_{cij}$) where $ij$ represents the subscripts of the spring constants and damping coefficients. From the FBD shown in Figure 5.2, equations of motion were derived:

$$\sum F_{M1} = F_{K12} + F_{c12} + F_c - F_{K01} - F_{c01} = m_1\ddot{x}_1$$  \hspace{1cm} (5.1)

$$\sum F_{M2} = F_{K23} + F_{c23} - F_{K12} - F_{c12} = m_2\ddot{x}_2$$  \hspace{1cm} (5.2)

$$\sum F_{M3} = -F_{K23} - F_{c23} = m_3\ddot{x}_3$$  \hspace{1cm} (5.3)

where

$$F_{K01} = K_{01}(x_1)$$  \hspace{1cm} $$F_{c01} = c_{01}(\dot{x}_1)$$

$$F_{K12} = K_{12}(x_2 - x_1)$$  \hspace{1cm} $$F_{c12} = c_{12}(\dot{x}_2 - \dot{x}_1)$$
The next condition took place at the instant the impact mass $M_3$ struck the seat, $x_2 = x_3 = OTD$. From this condition, a new FBD was developed and shown in Figure 5.3.

5.3 Initial Contact: Impact

Figure 5.3 - Diagram of CDTM: Condition 2 – Initial Contact: Impact

Figure 5.3 shows forces acting on the system under this condition and the forces can be divided into four categories: contact force ($F_c$), spring force ($F_{kij}$), damping force ($F_{cij}$), and impact force ($F_i$). From the FBD shown in Figure 5.3, equations of motion were derived:

\[
\begin{align*}
\sum F_{M1} &= F_{K12} + F_{c12} + F_c - F_{K01} - F_{c01} = m_1\ddot{x}_1 \\
\sum F_{M2} &= F_{K23} + F_{c23} - F_{K12} - F_{c12} = m_2\ddot{x}_2 \\
\sum F_{M3} &= F_i - F_{K23} - F_{c23} = m_3\ddot{x}_3
\end{align*}
\]

where

\[
\begin{align*}
F_{K01} &= K_{01}(x_1) \\
F_{K12} &= K_{12}(x_2 - x_1) \\
F_{K23} &= F_{K23\_pull} = K_{23}(x_3 - x_2) \\
F_{c01} &= c_{01}(\dot{x}_1) \\
F_{c12} &= c_{12}(\dot{x}_2 - \dot{x}_1) \\
F_{c23} &= c_{23}(\dot{x}_3 - \dot{x}_2)
\end{align*}
\]
\[ F_i = C^{(\gamma_{a+1})}\left\{ \frac{a+1}{2}m_2(v_1 - v_2)^2 \right\}^{(\gamma_{a+1})} \]  

With the above equations of motion, impact force was calculated and satisfied the impact modeling requirement. The missing component of the new dynamic model was the over-travel which is shown below.

### 5.4 Over-Travel

![Free Body Diagram: Condition 3](image)

Figure 5.4 shows the FBD for the third condition which applies when the impact mass \( M_3 \) and intermediate mass \( M_2 \) separates, \( x_2 < x_3 < OTD \). During the separation, the fixed preload force of the spring \( F_p \) tries to pull the two masses back together as shown in the FBD of Figure 5.4. The forces acting under this condition were divided into four categories; contact force \( (F_c) \), spring force \( (F_{kij}) \), damping force \( (F_{cij}) \), and preload force \( (F_p) \). From the above FBD, the last equations of motion for the CDTM were developed and shown below.
\[
\sum F_{M1} = F_{K12} + F_{c12} + F_c - F_{K01} - F_{e01} = m_1 \ddot{x}_1 \\
\sum F_{M2} = F_{K23} + F_{c23} + F_p - F_{K12} - F_{e12} = m_2 \ddot{x}_2 \\
\sum F_{M3} = F_{K03} + F_{e03} - F_{K23} - F_{c23} - F_p = m_3 \ddot{x}_3
\] (5.7) (5.8) (5.9)

where
\[
F_{K01} = K_{01}(x_1) \\
F_{e01} = e_{01}(\dot{x}_1) \\
F_{K12} = K_{12}(x_2 - x_1) \\
F_{e12} = e_{12}(\dot{x}_2 - \dot{x}_1) \\
F_{K23} = F_{K23_{pull}} = K_{23}(x_3 - x_2) \\
F_{c23} = c_{23}(\dot{x}_3 - \dot{x}_2) \\
F_{K03} = K_{03}(OTD - x_3) \\
F_{C03} = C_{03}(\dot{x}_3)
\]

### 5.5 Universal Equations of Motion

With these three conditions clearly understood, universal equations of motion were developed and presented below.

\[
\sum F_{M1} = F_{K12} + F_{c12} + F_c - F_{K01} - F_{e01} = m_1 \ddot{x}_1 \\
\sum F_{M2} = F_{K23} + F_{c23} + F_p - F_{K12} - F_{e12} = m_2 \ddot{x}_2 \\
\sum F_{M3} = F_{K03} + F_{e03} - F_{K23} - F_{c23} - F_p = m_3 \ddot{x}_3
\] (5.10) (5.11) (5.12)

where
\[
F_{K01} = K_{01}(x_1) \\
F_{e01} = e_{01}(\dot{x}_1) \\
F_{K12} = K_{12}(x_2 - x_1) \\
F_{e12} = e_{12}(\dot{x}_2 - \dot{x}_1) \\
F_{K23} = F_{K23_{pull}} = K_{23}(x_3 - x_2) \\
F_{c23} = c_{23}(\dot{x}_3 - \dot{x}_2) \\
F_{K03} = K_{03}(OTD - x_3) \\
F_{C03} = C_{03}(\dot{x}_3)
\]

If \( x_3 > OTD \)  
No Contact
\[
F_{K23} = K_{23_{push}}(x_3 - x_2) \\
F_{c23} = c_{23}(\dot{x}_3 - \dot{x}_2) \\
F_{K03} = 0 \\
F_{C03} = 0 \\
F_i = 0 \\
F_p = 0
\]

If \( x_3 = OTD \)  
Impact
\[
F_{K23} = K_{23_{pull}}(x_3 - x_2) \\
F_{c23} = c_{23}(\dot{x}_3 - \dot{x}_2) \\
F_{K03} = 0 \\
F_{C03} = 0 \\
F_i = C \left( \frac{a+1}{a} \right) \left\{ \frac{a+1}{2} m_2 \left( v_i - v_2 \right)^2 \right\}^{\frac{a+1}{a}} \\
F_p = 0
\]

If \( x_3 < OTD \)  
Over-travel
\[
F_{K23} = K_{23_{pull}}(x_3 - x_2) \\
F_{c23} = c_{23}(\dot{x}_3 - \dot{x}_2) \\
F_{K03} = K_{03}(OTD - x_3) \\
F_{C03} = C_{03}(\dot{x}_3) \\
F_i = 0 \\
F_p = \text{Calculated}
\]
Equations 5.10 through 5.12 represent the dynamic behaviors of the CDTM with impact and over-travel. Depending on the condition of $x_3$ with respect to $OTD$, various sets of equation can be applied and calculated. In order to obtain a numerical solution for these equations, CDTM’s masses, stiffness constants, damping coefficients, and impact parameters must be determined.
6 DETERMINING THE PARAMETERS OF THE CDTM

6.1 Lumped Masses Determination

To obtain an accurate dynamic model that represents the cam-follower system with impact and over-travel mechanism requires well-defined mass divisions. As seen in the previous section, the final dynamic model contains three masses. The first mass represents the follower mass, while the second and third masses represent the intermediate mass and impact masses, respectively.

![Figure 6.1 - CDTM with Impact and Over-Travel Mass Division](image)

Figure 6.1 - CDTM with Impact and Over-Travel Mass Division

Figure 6.1 shows the initial mass division where the abbreviations of the masses are shown in the legend of Figure 6.2.

The masses were lumped and transferred to the intermediate mass axis where the cart translates. Applying the material to these parts in CAD software, their masses were
easily determined. By combining the known masses as well as the lumped mass method described in Appendix A, it was possible to obtain the following lumped masses.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
<th>Weight (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{tre}$</td>
<td>Mass of Top Rod End</td>
<td>0.3854</td>
</tr>
<tr>
<td>$m_{bre}$</td>
<td>Mass of Bottom Rod End</td>
<td>0.1155</td>
</tr>
<tr>
<td>$m_{arl}$</td>
<td>Mass of Arm Rocker Left</td>
<td>0.0924</td>
</tr>
<tr>
<td>$m_{arr}$</td>
<td>Mass of Arm Rocker Right</td>
<td>0.1539</td>
</tr>
<tr>
<td>$m_{cr}$</td>
<td>Mass of Connecting Rod</td>
<td>0.7407</td>
</tr>
<tr>
<td>$m_{la}$</td>
<td>Mass of Link Arm</td>
<td>0.7949</td>
</tr>
</tbody>
</table>

Figure 6.2 - First Step Lumped Mass

Figure 6.2 shows the first step in lumping and relocating the masses where $m$, $k$, and $c$ represent the mass, stiffness constant, and damping coefficients, respectively. The link arm mass was lumped and relocated to the location where it connects with the connecting rod. The mass of the connecting rod was calculated but not translated in the first step. The right side of the arm rocker was lumped and relocated to the location where it connects with the connecting rod. The left side of the arm rocker was lumped and relocated to the intermediate mass’s axis of translation. Since the bottom rod
end components as well as the top rod end components were already located at the intermediate mass’s axis of translation, they were not relocated but lumped separately.

Combining the masses further to simplify the problem gives the following lumped masses.

\[
\begin{align*}
    m_1 &= m_{la} + m_{cr} = 1.5356 \text{ lb} \\
    m_2 &= m_{arr} = 0.1539 \text{ lb} \\
    m_3 &= m_{arl} + m_{bre} = 0.2079 \text{ lb} \\
    m_4 &= m_{tre} = 0.3854 \text{ lb}
\end{align*}
\]

**Figure 6.3 - Second Step Lumped Mass**

where \( m_1, m_2, m_3, \) and \( m_4 \) are the sum of link arm mass and connecting rod mass, arm rocker right mass, sum of arm rocker left and the bottom rod end components, and the top rod end components masses, respectively. These masses were combined but not relocated in the second step. In the last lumped mass step, the masses were combined and relocated to the intermediate mass’s axis of translation and is shown in Figure 6.4.

\[
\begin{align*}
    M_1 &= \frac{m_1 \times r_3^2}{r_4^2} = \frac{1.5356 \times 7.75^2}{2.375^2} = 16.35 \text{ lb} \\
    M_2 &= m_3 + \frac{m_2 \times r_3^2}{r_4^2} = 0.2079 + \frac{0.1539 \times 7.75^2}{2.375^2} = 1.638 \text{ lb} \\
    M_3 &= m_4 = 0.3854 \text{ lb}
\end{align*}
\]

**Figure 6.4 - Final Lumped Mass Model**
where $M_1$ is the equivalent of follower mass or the sum of link arm and connecting rod masses translated to the intermediate mass’s axis. $M_2$ is the equivalent of intermediate mass or the sum of arm rocker left and right as well as the bottom rod end components because these components do not experience the impact directly which is very similar to the intermediate mass in the industrial cam-follower system of interest. Lastly, $M_3$ is the equivalent of driving mass, impact mass, or the sum of the top rod end components because this mass disconnects from the rest of the system at the instant where impact occurs. Depending on the comparison of the simulated result to the experimental result, the masses’ divisions may be altered to obtain a more accurate comparison.

### 6.2 Lumped stiffness constants determination

Determining the stiffness constants of the components was the next step in creating a dynamic model. There are multiple ways of obtaining the stiffness constants of the components, including theoretical calculation through singularity function, finite element analysis using CAE software, and experimental calculation by applying a known load and obtaining the corresponding deflection. These methods have their own advantages and disadvantages. FEA software will yield an accurate result if the boundary conditions are properly applied but the simulation could take a substantial amount of time. In the meanwhile, the theoretical could easily be applied when appropriate simplifications are made. The deflection result obtained from the simplified theoretical method would be lower than that obtained through FEA methods because the simplified model would not contained the same features that made the actual component weaker. Neither of these methods is as accurate as an actual experiment. The only problem with the experiment would be the removable of every relevant component from the machine so that each one may be tested. Without proper boundary conditions, the experimental result would also be inaccurate. Once this dilemma arose, the only option that appeared viable would be to utilize two methods as a checking process. Therefore, finite element analysis and the simplified theoretical method were employed.
Applying the two methods mentioned in the previous paragraph, it was possible to obtain the stiffness constants of the CDTM. Detailed calculations of these stiffness constants can be seen in Appendix E.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
<th>Stiffness (lb/in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{bre _ push}$</td>
<td>Stiffness of Bottom Rod End (Push)</td>
<td>135,353</td>
</tr>
<tr>
<td>$k_{bre _ pull}$</td>
<td>Stiffness of Bottom Rod End (Pull)</td>
<td>224.8</td>
</tr>
<tr>
<td>$k_{arl}$</td>
<td>Stiffness of Arm Rocker Left</td>
<td>8,665</td>
</tr>
<tr>
<td>$k_{arr}$</td>
<td>Stiffness of Arm Rocker Right</td>
<td>850</td>
</tr>
<tr>
<td>$k_{cr}$</td>
<td>Stiffness of Connecting Rod</td>
<td>49,598</td>
</tr>
<tr>
<td>$k_{cs}$</td>
<td>Stiffness of Closure Spring</td>
<td>25.64</td>
</tr>
</tbody>
</table>

The values on the right of Figure 6.5 represent the stiffness constants shown in Figure 6.5. Each stiffness constant seen for bottom rod end components represent the stiffness as the bottom rod end is pushing the enclosure sleeve and stiffness when the
washer is pushing on the spring. Like its mass counter-part, the stiffness constants were combined in the same manner. The stiffness constant of the closure spring was combined with the stiffness constant of the connecting rod and transferred to the intermediate mass’s axis with the appropriate lever ratio. The stiffness of the arm rocker right was relocated to the intermediate mass’s axis while the stiffness constant of the arm rocker left was combined with the bottom rod end components stiffness constant. Combining and translating the mentioned stiffness constants created the following lumped parameters.

Calculated values

\[
K_{01} = \left( \frac{r_3}{r_4} \right)^2 \left( \frac{k_{cr} \times k_{cr}}{k_{cr} + k_{cr}} \right) = \left( \frac{7.75}{2.375} \right)^2 \left( \frac{25.64 \times 49,498}{25.64 + 49,498} \right) = 25.62 \text{ lb/in}
\]

\[
K_{12} = \left( \frac{r_3}{r_4} \right)^2 k_2 = \left( \frac{7.75}{2.375} \right)^2 850 = 9,051 \text{ lb/in}
\]

\[
K_{23} = \frac{k_{arl} \times k_{bre \_push}}{k_{arl} + k_{bre \_push}} = \frac{8,665 \times 135,353}{8,665 + 135,353} = 8,144 \text{ lb/in}
\]

\[
= \frac{k_{arl} \times k_{bre \_pull}}{k_{arl} + k_{bre \_pull}} = \frac{8,665 \times 224.8}{8,665 + 224.8} = 219 \text{ lb/in}
\]

Figure 6.6 - Final Lumped Stiffness Constant Model

After the lumped stiffness constants of the mechanism were determined, the approximate stiffness constant of the impact and over-travel system was calculated and combined with the stiffness constant of the hard-stop, as seen in Appendix C and Appendix E. It was found that the stiffness constant of the impact system \(K_{03}\) is 49,101 lb/in.

A summary of the simplifications made to the CDTM and their approximate values are shown in Figure 6.7.
**Simplified CDTM**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
<th>Summation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>Follower Mass</td>
<td>$m_{la} + m_{cr}$</td>
<td>16.351 lb</td>
</tr>
<tr>
<td>$M_2$</td>
<td>Intermediate mass</td>
<td>$m_{arr} + m_{arl} + m_{bre}$</td>
<td>1.6388 lb</td>
</tr>
<tr>
<td>$M_3$</td>
<td>Impact Mass</td>
<td>$m_{tre}$</td>
<td>0.3854 lb</td>
</tr>
<tr>
<td>$K_{01}$</td>
<td>Ground to $M_1$ Stiffness</td>
<td>$k_{ca} + k_{cr}$</td>
<td>25.62 lb/in</td>
</tr>
<tr>
<td>$K_{12}$</td>
<td>$M_1$ to $M_2$ Stiffness</td>
<td>$k_{cr} + k_{arr}$</td>
<td>9,051 lb/in</td>
</tr>
<tr>
<td>$K_{23}$</td>
<td>$M_2$ to $M_3$ Stiffness</td>
<td>$k_{arl} + k_{bre}$</td>
<td>8,144 lb/in</td>
</tr>
<tr>
<td>$K_{03}$</td>
<td>Ground to $M_3$ Stiffness</td>
<td>-</td>
<td>49,101 lb/in</td>
</tr>
<tr>
<td>OTD</td>
<td>Over-travel Distance</td>
<td>-</td>
<td>0.0625 in</td>
</tr>
<tr>
<td>Preload</td>
<td>$K_{23}$ (Die Spring) Preload</td>
<td>-</td>
<td>28.12 lb</td>
</tr>
</tbody>
</table>

Figure 6.7 - Simplified CDTM: Industrial 3-Mass 2-DOF with Calculated Parameters

Figure 6.7 shows the schematic diagram of a three-mass two-DOF model with the calculated physical properties. The missing values from the table in Figure 6.7 are the damping coefficients. However, it has been proven time and again that the typical
damping coefficient for cam-follower systems is approximately 6% of critical damping (Norton 2002). Therefore, it would be possible to utilize this value and adjust to better match the experimental result if needed. Otherwise, logarithmic decrement of dampen CDTM could be performed to obtain the damping coefficient.

7 SOLVING THE 3-MASS 2 DOF DYNAMIC MODEL

7.1 Solution Approaches

There are multiple methods in finding solutions of differential equations. The two methods to be discussed in this section are the state space and block diagram methods. State space uses a set of first order differential equations to describe the system of interest, which typically require the end-user to convert the system to conform to the required format. These consist of writing the equations of motion in terms of state variables and place these variables in state equation. Once the state equation has been finalized, the system must be placed in a state space form which is written in standard matrix format. This process may be easily performed for a one mass, one damper, and one spring system, it becomes rather complex for a higher degree of freedom model for an inexperienced user.

While state space is easy to use for a simple system, a block diagram is always simpler regardless of the complexity of the problem. As mentioned earlier, state space requires the end-user to transform the differential equations of motion into state space form. A block diagram does not require this process because the equations of motion may be replicated directly into the software. Because the state space method is more complex to utilize, block diagram method was employed and discussed in the following section.

7.1.1 Block diagram: Matlab and Simulink

Simulink, a graphical user interface tool for modeling, simulating, and analyzing dynamic systems, was suggested because it is currently being used by the sponsor.

---

7 Simulink – Product Description (http://www.mathworks.com/products/simulink/description1.html) (04/07/07)
Simulink is an extension of Matlab, a high-performance language for technical computing\textsuperscript{8}, which allows users to develop and alter the model with predefined and user-defined blocks.

The three main components of Simulink models are its inputs, mathematical operators, and outputs. The inputs of the cam vary depending on the specified profile; the user-defined input is the function that is necessary in modeling the cam-follower system. The user is able to export the theoretical $s, v, a$ cam functions from DYNA CAM and convert it into a Simulink-compatible file. After the input has been integrated in Simulink, mathematical operators were used to assist in the forming of the equations of motions. Once the solutions are found, results may be saved to a Matlab file, plotted, or saved into a workspace. These results can be converted to a Microsoft Excel-compatible format which allows further analysis.

Simulink allows the user to obtain the solution of the equations of motions very easily; the only task the user needs to accomplish is to develop the equations of motions and integrate them into Simulink. The initial conditions for the integration may be internal or external depending on the complexity of the equations. Prior to running the simulation, the type of solver to be used must be determined. The two main types of solvers are stiff and non-stiff. These two solvers are then divided into multiple versions developed by various authors. These solvers have different orders, accuracies, and methods. The basic method of determining whether a system is stiff or non-stiff is described in the next sections, while the comparison of the different solvers is described in a later section.

### 7.2 Solvers

#### 7.2.1 Stiff and non-stiff systems

Prior to solving the equations of motions of any system, it is necessary to determine whether the system is stiff or non-stiff. Using an incompatible solver will give inaccurate and unstable results, which will not correlate with experimental data. Inaccuracy and instability refer to the size of error for each step and growth in error over

\textsuperscript{8} Matlab – Product Description (http://www.mathworks.com/products/matlab/description1.html) (04/07/07)
subsequent steps, respectively. A system is stiff when the high and the low frequencies are widely separated and the high frequencies are highly damped. The system must also be stable, which means that none of its eigenvalues are positive. The system may have the ability to vibrate at a high frequency but be unable to, due to the association with the high damping which eliminates this frequency mode. A system may be stiff in one time period and non-stiff in another.

To determine whether the system is stiff or non-stiff, the stiffness ratio of the system must be determined. This value is determined by obtaining the highest inactive frequency and divided by the highest active frequency. Typically, a stiff system would have a stiffness ratio of 200 or higher while a non-stiff system would have a stiffness ratio less than 20. This means the non-stiff system’s higher frequency modes are active and the system actually vibrates at these frequencies. Since there is no known critical disadvantage of using a stiff-solver, most mechanical systems may be considered stiff. More information on this topic can be found in *Numerical Solution of Ordinary Differential Equations* by L.F. Shampine.

### 7.2.2 Available solvers

Since there are multiple solvers available in different software, choosing the most appropriate solver for each one was crucial to the analysis. Unfortunately, Pro/Mechanism only has one solver, Kane’s, while SolidWorks has GSTIFF, WSTIFF, and SI2 GSTIFF. Simulink has six solvers available for stiff and non-stiff systems. Understanding the advantages and disadvantages of each solver for the three software types is very important. After learning the lack of controllability of Pro/Mechanism, the software was removed for the tools used in analyzing the dynamic behaviors after impact. A table was created to compare the different solvers available.
<table>
<thead>
<tr>
<th>Program</th>
<th>Solver</th>
<th>System</th>
<th>Accuracy</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simulink</td>
<td>ODE45</td>
<td>Nonstiff</td>
<td>M</td>
<td>Dormand - Prince</td>
</tr>
<tr>
<td></td>
<td>ODE23</td>
<td>Nonstiff</td>
<td>L</td>
<td>Bogacki-Shampine</td>
</tr>
<tr>
<td></td>
<td>ODE113</td>
<td>Nonstiff</td>
<td>L to H</td>
<td>Adams</td>
</tr>
<tr>
<td></td>
<td>ODE15S</td>
<td>Stiff</td>
<td>L to M</td>
<td>NDF Gear</td>
</tr>
<tr>
<td></td>
<td>ODE23s</td>
<td>Stiff</td>
<td>L</td>
<td>Modified Rosenbrock</td>
</tr>
<tr>
<td></td>
<td>ODE23tb</td>
<td>Stiff</td>
<td>L</td>
<td>TR-BDF2</td>
</tr>
<tr>
<td>SolidWorks</td>
<td>GSTIFF</td>
<td>Stiff</td>
<td>H</td>
<td>Gear</td>
</tr>
<tr>
<td></td>
<td>WSTIFF</td>
<td>Stiff</td>
<td>H</td>
<td>Wielenga</td>
</tr>
<tr>
<td></td>
<td>SI2-GSTIFF</td>
<td>Stiff</td>
<td>H</td>
<td>Stablized-Index 2 (SI2) Gear</td>
</tr>
</tbody>
</table>

Table 4 - Simulink and SolidWorks Solvers Characteristics

As seen in Table 4, there are more solvers available in Simulink than in the others. This is to be expected because Simulink’s primary function is to analyze mathematical formulas, whereas SolidWorks’ COSMOSMotion was not designed with the same intent. Table 4 shows Simulink has both non-stiff and stiff solvers while COSMOSMotion only has the stiff solvers. Prior to using one of the above solvers, the user must determine if the system in question is a stiff or a non-stiff system, as discussed in the previous section. The ODE45 and ODE23 solvers are based on the one-step 4\textsuperscript{th} order and one-step 2\textsuperscript{nd} order explicit Runge-Kutta method, respectively. The disadvantage of this method is the medium and low accuracy, which results in a rapid computational time. For a higher accuracy simulation, ODE113 must be used. ODE113 solver is based on the multi-step variable order implicit Adams method. Although the computation time is longer than the Runge-Kutta method, the accuracy and efficiency is also higher when tolerance is very small. If a system is determined to be a stiff system, one of the following three stiff solvers must be used: ODE15s, ODE23s, or ODE23tb. ODE15s is based on multi-step variable order numerical differentiation formulas. The one-step 2\textsuperscript{nd} order modified Rosenbrock formula is represented by ODE23s. ODE23tb is based on the multi-step 2\textsuperscript{nd} order implicit Runge-Kutta formula, which uses a trapezoidal rule in the first stage and a backward differentiation formula in the second stage.
ODE15s is ideal for tight tolerances, whereas ODE23s and ODE23tb are more efficient and more accurate at crude tolerances.

GSTIFF is a solver based on backward differentiation formulas and Gear’s method. This method is a variable step and variable order solver with a maximum order of six. Variable step and variable order backward differentiation formulas and Wielenga method were represented in COSMOSMotion by WSTIFF. WSTIFF has a maximum order of six and the same advantages and disadvantages as GSTIFF, with one exception. The WSTIFF variable step does not incur any loss of accuracy, unlike GSTIFF. Both of these methods are able to determine the solution very fast, obtain accurate solution for displacements, and are very stable for most problems. Unfortunately, the velocities and accelerations have errors and fail at small step sizes. SI2 GSTIFF is essentially the same as GSTIFF, with the addition of stabilized index two. SI2 GSTIFF solves velocities and accelerations very accurately, is very stable at small step sizes, and is able to track the high frequency oscillations accurately. The problem with this method is the slow calculation time and the input velocity function must be continuous or differentiable.

The scope of this thesis is not the solvers per se, but the accuracy of the solution. Therefore, an attempt to obtain more information and understand these solvers would be outside of the scope and was not considered. However, there is voluminous literature that contains the details of these solvers, such as *The Simultaneous Solution of Differential Algebraic Systems* and *Numerical Initial Value Problems in Ordinary Differential Equations* by C.W. Gear for the GSTIFF solver.
7.3 Simulink: 3-Mass MDOF Model

Figure 7.1 - Simulink's 3-Mass 2-DOF Industrial Model

Figure 7.1 shows the Simulink sub-system with the four main inputs which are the DYNACAM input, physical properties, impact parameters, and experimental results. As seen in Figure 7.1, the DYNACAM input consists of the theoretical position, velocity, and acceleration of cam roller follower. The physical properties consist of the lumped masses, stiffness constants, and damping coefficients. Impact parameters consist of the previously determined slope, proportionality constant, specific over-travel distance, and
the preload force in the over-travel mechanism. Lastly, the experimental results consist of the experimentally obtained position, velocity, acceleration, and force. Damping calculation, over-travel force calculation, and impact force calculation sub-sub-systems are located within the sub-system. These sub-sub-systems are created within the initial sub-system to minimize the confusion and to better organize the available space in Simulink as seen in Figure 7.2.

Figure 7.2 - Simulink's Sub-System of 3-Mass 2-DOF Industrial Model
The main sub-system was divided into five sub-sections, as shown in Figure 7.2, which are the inputs, damping calculation, forces calculation, equations of motion, and results which are also shown in Figure 7.3 through Figure 7.7. Multiple Go to and From blocks were used in these sub-sections. As seen in Figure 7.3, the data from the physical properties inputs were inserted into Go to blocks. Not only the data from the inputs were directed to Go to blocks but also the simulated position, velocity, and acceleration of mass 2 and 3, as seen in Figure 7.7. This was done to eliminate the possibility of data lines intersecting with each other and was performed only for the variables which were used at multiple locations. After the data were redirected into the Go to block, they were called by the From block as seen in the two equations of motion, Figure 7.7, as well as the results, Figure 7.5.

The damping calculation in sub-section 2 was expanded and, shown in Figure 7.4, was created to group the damping calculation together. These calculations are simply:

\[
c_1 = 2m_1\zeta_1 \sqrt{\frac{k_1}{m_1}}, \quad c_2 = 2m_2\zeta_2 \sqrt{\frac{k_2}{m_2}}, \quad c_3 = 2m_3\zeta_3 \sqrt{\frac{k_3}{m_3}}
\]

The results were directed into the Go to blocks and called through From blocks as needed in the sub-section 4, equations of motion.

The more interesting area is sub-section 3, shown in Figure 7.6. In this section, the calculations of the forces experienced by the force transducer were calculated. These forces are the preload force, over-travel force, and the impact force. The preload force is a step function which is either zero or 125.1 N. The condition that triggers the switch is the position of the impact mass relative to the over-travel distance. If the position of
impact mass is less than or equal to the over-travel distance, the preload force component experienced by the force transducer is 125.1 N, otherwise it is zero.
Figure 7.8 - Over-Travel Force Calculation for 3-Mass 2-DOF Model

Figure 7.8 shows the over-travel force calculation which was filtered by the two conditions. If either the force is less than zero and/or the position of mass 3 is greater than the over-travel distance, the output equals zero; otherwise, the output is calculated on the left.

Figure 7.9 - Impact Force Calculation for 3-Mass 2-DOF Model

The last and most important component of the force experienced by the force transducer is the impact force calculation. By inserting the proportionality constant, slope, impact mass, and impact velocity, impact force was calculated through the use of the common velocity approximation method. Therefore, Figure 7.9 is the graphical representation of the following equation.
\[ F_i = \left(1.11 \times 10^7\right) \left(2.425 \frac{m_2}{2} \frac{v_i}{v_{2,425}} \right)^{\frac{1}{2}} \left(1.425 \frac{x_2}{v_{2,425}} \right)^{\frac{1}{2}} \]  

(7.1)

With these three forces components, the simulated force experienced by the force transducer can be obtained and compared to the experimental data.

In Section 4, the graphical representation for the equations of motion is presented. Very little difference can be seen in the upper equation of motion which calculates the position, velocity, and acceleration of mass 2. The same cannot be said for the equation of motion for mass 3. Looking closely, it is seen that the upper left hand side of the bottom graphical representation for the equation of motion of mass 3 contains a stiffness constant switch block. This switch block switches the stiffness constant of the over-travel mechanism depending on the condition of the mass 3 with respect to the over-travel distance. If the position of mass 3, \( x_3 \), is less than or equal to the over-travel distance, the stiffness constant equals the lower stiffness constant or 38,353 N/m; otherwise, the stiffness constant equals 1,426,233 N/m.

The other interesting aspect of this model is the inclusion of the seat stiffness, \( k_{03} \), which had already been mentioned earlier in this section. The seat stiffness is also a step function that is either zero or 8,597,740 N/m depending on the same condition as the preload force.

After the creations of the equations of motion as well as the forces were completed, the results blocks were created and are located in Section 5 of Figure 7.2. The result section includes the theoretical inputs, comparison between the experimental and the simulated result of mass 2, the comparison among the experimental and simulated results of mass 2 and 3, the simulated result of the force experienced by the force transducer, the different between simulated mass 2 and 3, and the simulated results of mass 2 and 3 saved into Matlab files. The only task left was to obtain the experimental data from the CDTM with the impact and over-travel events and compare the results.
8 RESULTS

8.1 Simulink Results

Figure 8.1 shows the schematic diagram of the intermediate mass, $M_2$, and the impact mass, $M_3$; their simulated displacement comparison is shown in Figure 8.2. These two displacement curves match well with the exception of the over-travel periods which are labeled A, B, and C in Figure 8.2. This was expected because these masses have relative motion during the over-travel periods, when they separate. During the separations, the impact mass exerts a large force on the force transducer, twice per cycle. Once the vibration from impact subsides, the two masses move as one. Before events 1 and 3 of Figure 8.2, $M_2$ is moving toward $M_3$, while $M_3$ is pressed against the force transducer, as shown in Figure 8.3.

![Figure 8.1 - CDTM Equivalent 3-Mass 2-DOF Schematic Diagram](image-url)
Masses $M_2$ and $M_3$ collide at points 1 and 3 of Figure 8.2. The force transducer does not record these impact events because the force is not acting on the transducer, but on the enclosure sleeve and the bottom rod end at point of impact 1, as depicted in Figure 8.4.

Before points 2 and 4 of Figure 8.2, the two masses move together and separate at these points. At points 2 and 4, the impact mass $M_3$ strikes the force transducer while the intermediate mass $M_2$ continues on its path. This is labeled “point of impact 2” in Figure 8.4. These two impacts as well as the over-travel periods are recorded by the force transducer as seen in Figure 8.10.
Figure 8.3 - Impact Mechanism: Over-Travel

Figure 8.4 – Impact Mechanism: Initial Contact

Figure 8.5 – Impact Mechanism: Zero Force
Figure 8.6 compares the simulated velocities of masses $M_2$ and $M_3$. These two curves correlate well but differ during the over-travel periods, labeled A, B, and C in Figure 8.6. After the two masses come into contact and attain the same velocity, there are few differences until the next impact occurs. The first and third impacts occur when $M_2$ strikes $M_3$, while the second and fourth impacts occurred when the impact mass $M_3$ strikes the force transducer, labeled seat in Figure 8.1.

Note in Figure 8.6 that $M_3$ rings out faster than $M_2$ during periods B and C. This is explained by their difference in natural frequencies when separated. The stiffness constant of the seat, $K_{03}$, is approximately six times greater than the stiffness constant of the over-travel mechanism, $K_{23}$ push, and it is assumed that the two masses have the same damping ratio. Due to the six times higher stiffness, the natural frequency of the impact system is also $\sqrt{6}$ or 2.45 times greater than intermediate mass’s system. As a result, the period of the impact system, $\frac{1}{\omega_n}$, is $\frac{1}{\sqrt{6}}$ or approximately 40% of the intermediate mass’s period. To ensure that the stated assumption is valid, the vibration after the second impact for $M_2$ and $M_3$ were normalized and plotted as shown in Figure 8.7 and Figure 8.8.
In Figure 8.7 and Figure 8.8, two data points were marked and labeled with the maximum amplitude of vibration, the point of ten percent of the maximum after the second impact, and the time at which these events occurred. These data points were used to determine a measure of the impact period for $M_2$ and $M_3$. $M_2$ took 0.02136s (0.20330s
– 0.18194s) for the magnitude of vibration to reduce to 10% of the maximum and \( M_3 \) took 0.007639s (0.18646s – 0.17882s) for the same reduction to occur. Dividing these two periods gives the ratios of their ringouts.

\[
\frac{M_3 \text{ vibration period relative to } M_2} = \frac{0.007639}{0.2136} \approx 36\%
\]

Although the above result, 36%, does not match the predicted relative vibration period of 40%, this was expected because it was stated that the stiffness of \( M_3 \) after the second impact was approximately six times greater that of \( M_2 \). Also, the marked data points were not exact because of round-off errors. In general, the theoretical prediction matched the approximate value reasonable well.

![Simulated Acceleration Comparison](image)

**Figure 8.9 - Simulated Acceleration Comparison of \( m_2 \) and \( m_3 \)**

Figure 8.9 shows the simulated accelerations of masses \( M_2 \) and \( M_3 \). As expected, the magnitude of the vibration of \( M_3 \) is 4 to 5 times greater than that of \( M_2 \) because that is the location where impacts occur. Again, the first and third impact events dampen out much faster than that of the second and fourth because the stiffness constants of the system after the first and third impact are lower. The absolute maximum simulated acceleration of \( M_2 \) and \( M_3 \) are 167 m/s\(^2\) and 841 m/s\(^2\), or 13 and 67 times greater than the theoretical acceleration at the same location. The upward shifted in magnitude of vibration is a result of the collision of \( M_2 \) and \( M_3 \). Because of this, the maximum positive
acceleration is three times greater than the maximum negative acceleration at the first and third impacts. Because the sensor is located on the intermediate mass, $M_2$, its acceleration will be compared to the experimental results.

Figure 8.10 shows the simulated force as a result of $M_3$ striking the force transducer. It also represents the sum of three forces which are impact, preload, and over-travel forces. Preload force is a step function, which is either 0 or 125.1 lb. However, the over-travel force is a function of the displacement; its value equals the relative displacement, $x_3-x_2$, times the spring constant of 225 lb/in. Lastly, the impact force is a function of the velocity at impact and other factors.
8.2 Experimental Results with Impact and Over-travel

Figure 8.11 shows the location of the sensors used in the experiment with impact and over-travel. The four sensors used in the experiments are displacement, velocity, acceleration, and force which are shown in Figure 8.11 as LVDT, LVT, Accelerometer, and Force Transducer, respectively. It is important for the reader to observe that the displacement sensor (LVDT) and the velocity sensor (LVT) are connected and are used to measure data from the link arm. Due to their location, their output cannot be used to compare to the simulated data which are predicting the displacement and velocity at the intermediate mass. However, the acceleration and the force data obtained from experiment can be used to compare to the simulated result because accelerometer and force transducer were mounted at the intermediate mass and hard-stop, respectively.

Figure 8.12 through Figure 8.15 show the experimental displacement, velocity, acceleration, and force data with impact and over-travel, respectively.
Figure 8.12 shows the experimental displacement data which includes impact and over-travel events. Unfortunately, the displacement sensor (LVDT) was not directly connected to the intermediate mass but to the link arm, thus effect of impact and over-travel events were not noticeable. Although, this result cannot be compared to the simulated result, it is still critical to understand and analyze its output.

By closely examining the displacement plot, it can be said that the cam used in this experiment was not manufactured properly because of the inconsistencies of the dwells. As seen in Figure 8.12, the first top dwell appeared to be much flatter than the second top dwell. The second top dwell also started higher and tapered off by the end of the dwell which is undesirable. Also, the second bottom dwell did not reach the intended minimum displacement of zero unlike the first bottom dwell. From these observations, it is possible to predict that the simulated results will not correlate with the experimental precisely due to these imperfections.
Figure 8.13 shows the experimental velocity data with impact and over-travel events. Again, the velocity sensor (LVT) was connected to the link arm instead of the intermediate mass, thus it cannot be used to compare to the simulated velocity. However, the above experimental velocity presents noticeable effects of impact events which are the result of the second and fourth impacts. Second and fourth impacts are impact events which occur due to the impact mass $M_3$ strikes the hard-stop. Even though the second and fourth impacts events were relatively noticeable, the first and third impacts were not noticeable because the backlashes of first and third impact were relatively small compared to that of the second and fourth as seen in the experimental acceleration plot, Figure 8.14.

Another important and very pronounce events were the vibration due to the splits in the cam, as labeled in Figure 8.13. These splits in the cam were not intended in the original design but because of assembly process constrains, the cam was manufactured in two pieces. These vibrations are also apparent in the experimental acceleration plot shown in Figure 8.14.
Figure 8.14 shows the experimental acceleration data with impact and over-travel events. Because the accelerometer was mounted on the intermediate mass, as seen in Figure 8.11, the data obtained was readily used to compare to the simulated result which is presented in the next section. The first observation made was the magnitude of the vibration due to the splits in the cam. One would expect the vibration to be more pronounce in the acceleration result than the velocity result. However, the vibrations due to splits in the cam were more pronounce in the velocity plot because of the LVT was mounted inches away from the cam. At the same time, the accelerometer was mounted on the intermediate mass, thus the LVT was able to obtain a clear and undamped signal while the accelerometer received damped signal.

Aside from the splits in the cam which may be considered irrelevant to the research, the relative magnitudes of the vibration of the acceleration were as expected. The first and third impacts, labeled in Figure 8.14, were much lower and damped out faster than the second and fourth impacts. The absolute maximum acceleration of the first and third impacts is approximately 100 m/s while the second and fourth is 200 m/s. These results correlate well with the simulation results which are presented in the next section.
Figure 8.15 - Experimental Force Data with Impact and Over-travel Events

Figure 8.15 shows the experimental force obtained through the force transducer with the time scale shifted to eliminate data cut-off which is apparent in the simulated result, Figure 8.10. Since the force transducer is mounted on the hard-stop which is the location where the second and fourth impact occurs, the obtained data is the sum of three forces; preload, impact, and over-travel. The shape of the experimental result is very similar to that of the simulated result. However, the maximum force during the first period is slightly higher than that of the second period. This is not an expected result because the first and second period should be identical according to the intended design specification. However, this phenomenon was traced back to the inconsistencies of the cam which was very pronounced in the experimental displacement data, Figure 8.12.

In order to understand this phenomenon, the experimental displacement data as well as the experimental force data were normalized, shifted, and presented in Figure 8.16 to compare and present the relationship between them.
As mentioned earlier in this section, the cam profile itself contains imperfections. From these imperfections, the force output is also inconsistency because one component of the force, over-travel force, is a function of displacement, \( F_{x_3} = K_{o3} (OTD - x_3) \) when \( x_3 < OTD \). With this equation in mind, the experimental displacement function is inserted into the equation and it is apparent that the second bottom dwell would exert less force because the displacement during this period is greater than the first bottom dwell.

After the experimental data were obtained and understood, they were compared to the simulated data obtain from the dynamic model created in Simulink. These comparisons are presented in the next section.
8.3 Experimental and Simulated Results Comparisons

Figure 8.17 compares the experimental and simulated acceleration of $M_2$. The maximum magnitudes of the first and third impacts match relatively well, even though the shapes after impact do not. The main reason for this discrepancy is the lower degree-of-freedom of the simulated system versus the real system. By reducing DOF, higher modal characteristics of the links were excluded from the model. The maximum magnitudes of the simulated acceleration for the second and fourth impacts were 1.84 (at point A) or 1.32 times smaller (at point B) than the experimental acceleration, respectively. Also, the vibrations during the dwells, at 0.10 and 0.35 sec, were the result of the splits in the cam, which were not included in the simulation.

Since the simulated data was superimposed on the experimental data and may be unclear, these two results were separated and presented in Figure 8.18 and Figure 8.19.
Figure 8.18 - Simulated Acceleration of Intermediate mass ($M_2$)

Figure 8.19 - Experimental Acceleration of Intermediate mass ($M_2$)
As seen in Figure 8.18 and Figure 8.19, the simulated first and third impacts yield higher acceleration than the experimental result. However, the same cannot be said for the second and fourth impacts because the simulated results predicted lower absolute maximum values. Even though there are slight discrepancies, the simulated acceleration still correlated reasonably well with the experimental acceleration.

![Impact and Over-travel Force Comparison](image)

**Figure 8.20 - Experimental vs. Simulated Impact and Over-travel Force**

Figure 8.20 compares the experimental force to the simulated force. Since, this is an approximation; the simulated results do not match the experimental results perfectly. It is impossible to obtain a perfect step function in a physical experiment, which means the preload force in the experiment would have finite slope while the simulated force has infinite slope. The impact component does not match precisely either, because the impact force calculation method employed is an approximation which resulted in inaccurate estimates. The simulated over-travel component fails to correlate with the experimental results because the simulation does not include the transducer’s discharge time constant. Aside from these minor differences, it is observed that the maximum simulated force overestimated the maximum experimental force by 2 lb, or 1.3%, and the shapes of the functions correlate reasonably well.
### 8.4 Simulated No Impact and Impact Comparisons

From the correlation found in the previous section, it was determined that the simulated model which included impact and over-travel events was a good representation of the actual system. In order to justify the work involved in creating this model, dynamic model with no impact was compared to a dynamic model with impact and over-travel events. Depending on the comparisons, it would be possible to determine whether impact and over-travel events are needed for future dynamic model. Therefore, displacement, velocity, and acceleration were compared as seen in Figure 8.21 through Figure 8.26.

**Figure 8.21 - Mass 2 Simulated Displacement: No Impact vs. Impact**

Figure 8.21 compares the displacement of dynamic models with impact and no impact events. Although, the differences are minimal during the over-travel periods labeled A, B, and C in Figure 8.21, these differences are amplified in the velocity and acceleration which are presented later in this section. However, prior to analyzing the differences in velocity and acceleration, it is critical to observe the percent differences between these two simulated models.
Figure 8.22 shows the displacement percent differences between the impact and no impact models. Data seen in Figure 8.22 was obtained by utilizing the following equation:

\[
\frac{\text{dis}_{\text{imp}} - \text{dis}_{\text{no_imp}}}{\text{dis}_{\text{imp-max}}}
\]

where \( \text{dis}_{\text{imp}} \), \( \text{dis}_{\text{no_imp}} \), and \( \text{dis}_{\text{imp-max}} \) are the simulated displacement with impact, simulated displacement without impact, and the absolute maximum displacement with impact, respectively, to determine the percent different between the two displacements. The highest percentage differences, approximately 2%, occurred during the over-travel periods labeled A, B, and C which were not included in the no impact dynamic model. From the highest percentage difference observed in Figure 8.22, it may be possible to argue that two percent different is insignificant and the impact and over-travel may be neglected from future dynamic models. This conclusion may be appropriate if displacement is the only parameter of concerned. However, that is not the case in most situations and velocity and acceleration must be considered.
Simulated Velocity Comparison

Figure 8.23 - Mass 2 Simulated Velocity: No Impact vs. Impact

Simulated Velocity Percent Difference

Figure 8.24 - Mass 2 Simulated Velocity: No Impact vs. Impact % Difference
Figure 8.23 and Figure 8.24 compares the velocity and presents the percent differences of dynamic models with impact and no impact events. The differences seen in Figure 8.23 are much more pronounce than those seen in the displacement comparison. These differences occurred immediately after the impacts labeled 1, 2, 3, and 4 in Figure 8.23. The differences immediately after the 1\textsuperscript{st}, 2\textsuperscript{nd}, 3\textsuperscript{rd}, and 4\textsuperscript{th} impacts are 0.09 m/s, 0.11 m/s, 0.09 m/s, and 0.11 m/s respectively. However, a better comparison of the two data was obtained by utilizing the following equation:

$$\frac{vel_{imp} - vel_{no\_imp}}{vel_{imp\_max}}$$

where $vel_{imp}$, $vel_{no\_imp}$, and $vel_{imp\_max}$ are the simulated velocity with impact, simulated velocity without impact, and the absolute maximum velocity with impact, respectively, to determine the percent different between the two velocities. The absolute maximum percent different observed in this figure is approximately 40\%. This value certainly is much more significant than the 2\% difference observed in the displacement comparison. Thus, impact and over-travel events should not be neglected based on the forty percent difference observed from the velocity comparison.

![Simulated Acceleration Comparison](image)

Figure 8.25 – Mass 2 Simulated Acceleration: No Impact vs. Impact
Figure 8.25 and Figure 8.26 compares the acceleration and presents the percent differences of dynamic models with impact and no impact events. The differences seen in Figure 8.25 are much more pronounced relative to velocity or displacement comparisons. Again, these differences occurred immediately after the impacts labeled 1, 2, 3, and 4 in Figure 8.25 and are well over four times the theoretical accelerations. By using the following equation:

\[
\frac{acc_{imp} - acc_{no\_imp}}{acc_{imp-max}}
\]

where \( acc_{imp} \), \( acc_{no\_imp} \), and \( acc_{imp-max} \) are the simulated acceleration with impact, simulated acceleration without impact, and the absolute maximum acceleration with impact, respectively, to determine the percent different between the two accelerations. The absolute maximum percent different observed in this figure is approximately 100%, the highest possible. Therefore, the acceleration comparisons had essentially reinforced the hypothesis that impact and over-travel events cannot be neglected if a reasonably accurate dynamic model was to be obtained.
SUMMARY AND CONCLUSIONS

Based on the correlations of acceleration and force described in section 8.3, a successful experimental investigation and modeling of impact in an over-travel mechanism was performed. Impact and over-travel mechanisms were designed and manufactured to create measurable forces when impact occurred. A relatively accurate 3-DOF dynamic model was created and simulations run to predict the dynamic behavior of the assembly machine and associated forces. The model, consisting of three second-order differential equations, was driven by the cam’s input function. This model included impact and over-travel forces which were not included in other industrial cam-follower system models found during an extensive literature search.

During modeling, multiple methods of approximating the impact force were discovered. Investigation of the best impact force approximation was performed by experimentally measuring impact force between a spherical steel ball and a flat force transducer and comparing it to the two approximation methods. The common velocity method overestimated the impact forces by 10% to 25% whereas the energy method underestimated the impact forces by 35% to 40%.

Another discovery was the inadequacy of the two-mass SDOF model for a cam-follower system with impact and over-travel. Through an extensive literature search, it was determined that the dynamic model for the cam-follower must consist of at least two masses. One of the masses allowed an approximation of the contact force between cam and follower while the other mass was used to predict dynamic behavior. By including an impact and over-travel event, a third mass was added that represented a striking mass that was used to determine the impact force.

Use of this impact model for an industrial cam-follower system can guide mechanical engineers through the designing stage of assembly machines with impact and over-travel more efficiently. This model will allow the elimination of expensive and time-consuming full modeling methods, which will reduce machine development costs as well as development time.
10 RECOMMENDATIONS

The research completed is only the initial step in creating a dynamic model of a cam-follower system with an impact and over-travel event. Although the correlations found in this research are very promising, further development should be done to obtain a superior model. The new model will more accurately predict the dynamic behavior and impact forces. In order to obtain an improved model, the following recommendations are proposed:

1. Create a higher degree-of-freedom model. Even though in most cases the first mode of vibration contributed significantly more than other vibration modes to the amplitude of vibration, a better correlation will be produced with a higher degree-of-freedom model. This was proven by Seidlitz with a twenty-one degree-of-freedom model. Therefore, an optimization study may be conducted to determine the optimal degree-of-freedom versus development time. These results indicate that a higher degree-of-freedom model can be created and simulated to yield more accurate results.

2. Implement the wave method to better approximate the impact force. Johnson’s common velocity method (Johnson 1958) represents the impact force more accurately than the energy method (Burr 1982). The accuracy of the wave method was not examined in this study. Therefore, this investigation should be performed to determine whether the accuracy advantage outweighs the benefits of the other methods. Once this is known, the wave method may be included in the dynamic model.

3. Change the materials of the impact parts. Impact force depends mainly on the driving and driven masses, impact velocity, and material properties. By researching ideal pairs of materials to be used for the impact, a minimum impact force output as well as minimum wear from the impacts can be obtained.
REFERENCES


Appendix A: Dynamic Modeling Techniques

In this section, fundamental dynamic modeling techniques will be discussed such as lumped mass calculation for rotating and translating parts, determination of stiffness constants of the parts in bending, tension, or compression, and damping coefficient methods. These physical data are needed in equations of motion which describe the dynamic behavior of the system such as Figure A.1. As stated in Norton (2002), for the lumped mass of a rigid body to be dynamically equivalent to the original body, three conditions must be satisfied:

1. *The mass of the model must equal that of the original body.*
2. *The center of gravity must be in the same location as the original body.*
3. *The mass moment of inertia must equal that of the original body.*

Mass

The two types of motions that most parts or sub-assemblies undergo are translation and rotation. There is usually little or no complex motion. When complex motion is present, it is typically simplified to either rotation or translation to minimize the complexity of the problem.

Masses of the translating parts are easily obtained by multiplying the volume of the part, $V$, by the mass density, $\rho$, or computed through CAD software. The manual calculation method may be used if the part has simple geometry or has been simplified. Whenever simplifications are made, inaccuracies occur which amplify the error of the simulation. Since the parts have already been created in CAD software, their masses can be calculated with ease and with accuracy superior to that of manual calculations.

Masses in rotations require more computation to be lumped. The calculation may be performed manually by simplifying the part and calculating the moment of inertia with respect to the axis of interest. Again, inaccuracies from the simplification are amplified in the simulation. The easiest calculation method without inaccuracies is to use CAD
software. One can specify the axis of interest and have the CAD software calculate the mass moment of inertia of the part with respect to that axis. Since the mass moment of inertia equals:

\[ I_{zz} = M \times r^2 \]  

(A.1)

where \( I_{zz} \) is the mass moment of inertia of the part with respect to the \( z \)-axis, \( M \) is the lumped mass of the part, and \( r \) is the radius of the axis of rotation to the point of interest, it is possible to obtain the mass of the rotating part by calculating \( M = \frac{I_{xx}}{r^2} \).

**Spring rate**

The spring rate or stiffness constant of each part has to be computed before a dynamic model of the cam follower system can be obtained. These parts will have to be constrained the same way as it would on the machine. The part may be removed from the machine and tested. The testing procedure includes constraining the part in a similar manner as the machine, placing a known weight or force on the part as it would experience in the machine, and measuring the displacement of the part. Since we know that the spring rate or stiffness constant of a part can be calculated from the equation

\[ F = k \cdot \delta \]  

(A.2)

where \( F \) is the known force in Newtons (N) or pound force (lb), \( k \) is the stiffness constant in Newtons per meter \( (N/m) \) or pounds per inch \( (lb/in) \), and \( \delta \) is the displacement in meters or inches. The above procedure may not be applicable in most cases because the process requires the removal of the part from the production machine, which would hinder the production processes.

If the above method cannot be employed, an approximate calculation could be performed. Most parts in any mechanism deflect in either tension/compression or bending. If the part is in either tension or compression, the utilization of the following equation would give the appropriate stiffness constant:

\[ \delta = \frac{FL}{AE} \Rightarrow k = \frac{F}{\delta} = \frac{AE}{L} \]  

(A.3)

where \( L, A, E \) are the length, cross-section area, and modulus of elasticity of the member in tension/compression. As for the parts in bending, a singularity function may be applied depending on the boundary conditions of the parts.
Another solution that is utilized frequently is the Finite Element Analysis (FEA). FEA is a process that requires one to model the part in CAD software. Once the part is created, assigning the material to the part is the next critical step. The part is then divided into many elements. The number of elements can be specified by the designer. The designer must constrain the model the same way that it would be constrained in the machine using the given constraints available in the program. After the part is fully constrained, a force of the same magnitude should be applied to the part at the same location as the part would experience in the machine. The CAD software will be able to output a displacement fringe and the designer will be able to obtain the displacement and use the above equation to solve for the spring rate of the part.

**Damping**

Damping coefficient was said to be the hardest parameter to model (Norton 2002). That is because there are multiple types of damping, which are coulomb damping, viscous damping, and quadratic damping, as seen in Figure A.2.

![Figure A.2 - Column, Viscous, Quadratic, and Approximation Damping (Norton 2002)](image)

Coulomb, viscous, and quadratic damping are the frictions that occur when two surfaces rub together, when the lubricant is sheared, and when an object moves through a viscous medium, respectively. These three types of damping were combined into one linear approximated damping. To determine the damping coefficient, an experiment must be performed on the machine of interest. A piezoelectric accelerometer will be attached at the output, where the tooling meets the part. The method of attachment includes stud, cement/wax, and magnet. The preferred attachment would be stud, because it will give the highest frequency response of 7 kHz. A hammer with a built-in piezoelectric transducer will be used to induce a blow near the location where the cam
and follower meets. The data obtained from the piezoelectric accelerometer and the hammer readings will be transferred to an oscilloscope. The data from the oscilloscope is then analyzed for the values of the first two peaks using the following equation.

\[ \delta = \ln \left( \frac{x_1}{x_2} \right) \]  

(A.4)

where \( x_1 \) is the first peak value, \( x_2 \) is the second peak value, and \( \delta \) is the logarithmic decrement. The damping ratio can be calculated using:

\[ \zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \]  

(A.5)

where \( \zeta \) is the critical damping ratio.

**Combining the parameters**

After one has obtained the masses, stiffness constants, and damping coefficients, it is now possible to reduce their complexity further by combining them together into a lower number of DOF. This is possible by utilizing the lever and gear ratio.

**Lever and gear ratio**

![Diagram of lever and gear ratio](image)

Figure A.3 - Lever and Gear Ratios Sample

Figure A. shows two masses, two springs, two dampers, and two different distances from the point of rotation. First, the assumption that the springs and dampers are acting at the center of mass 1 \( (m_1) \) and mass 2 \( (m_2) \) has to be made. Also, the distances \( a \) and \( b \) are the distance from center of \( m_1 \) and \( m_2 \) to the pivot point, respectively. In this example, we will be transferring the \( m_2, k_2, c_2 \) to the left side.
Transferring the mass from one side to another requires the system to maintain its kinetic energy. Therefore, the equation that will be used is the kinetic energy equation:

\[
\frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_{eq2} v_1^2 \quad (A.6)
\]

where \( v_2 \) is the velocity of \( m_2 \), \( v_1 \) is the velocity of \( m_1 \), \( m_{eq2} \) is the \( m_2 \) equivalent when transferred to the left side. The velocities of both masses are:

\[
v_1 = \left( \frac{a}{b} \right) v_2 \quad (A.7)
\]

by substituting and manipulating the velocity equation into the kinetic energy equation. The following equation was obtained for the equivalent mass after it was transferred to the left side:

\[
m_{eq2} = \left( \frac{b}{a} \right)^2 m_2 \quad (A.8)
\]

After the mass has been transferred to the left side of the equation, it is possible to combine the masses by simply adding the two together:

\[
m_f = m_1 + m_{eq2} = m_1 + \left( \frac{b}{a} \right)^2 m_2 \quad (A.9)
\]

where \( m_f \) is the combined mass or final mass of the system.

The same method can be applied for the springs by utilizing the kinetic energy equation of the spring:

\[
\frac{1}{2} k_2 x_2^2 = \frac{1}{2} k_{eq2} x_1^2 \quad (A.10)
\]

where \( x_2 \) and \( x_1 \) are the displacements of \( m_2 \) and \( m_1 \) in the direction that extends or compresses the spring \( k_2 \) and \( k_1 \), respectively, and \( k_{eq2} \) is the \( k_2 \) equivalent when it is transferred over to the left side of the system. Employing a similar method as shown above, one will be able to obtain the equivalent spring rate after transferring it to the left side:
Combining the spring rate after transferring to the left side is a little more complex than combining the mass but can be done by using the following methods. If the force passing through each spring is the same but has a different displacement, as shown in Figure A., then the springs are in series and maybe combined using the following equation:

\[ k_{\text{eff}} = \frac{1}{k_1 + k_2} \]  \hspace{2cm} (A.12)

As for the case where the forces passing through each spring are different but the displacement of each spring is the same, that means the springs are in parallel, as shown in Figure A., and the effective spring constant would be the sum of the spring constants.

\[ k_{\text{eff}} = k_1 + k_2 \]  \hspace{2cm} (A.13)

In the case shown above, both springs would have the same displacement, which means the effective spring rate of the system would be:
\[ k_{\text{eff}} = k_1 + k_{\text{eq2}} = k_1 + \left( \frac{b}{a} \right)^2 k_2 \] \hspace{1cm} (A.14)

Transferring the damping coefficient to the left side is accomplished by using the force equation
\[ F_{\text{di}} \times a = F_{\text{dr}} \times b \] \hspace{1cm} (A.15)
where \( F_{\text{di}} \) is the damping force on the left and \( F_{\text{dr}} \) is the damping force on the right. Since it is known that the damping force is the product of the damping coefficient times the velocity and after manipulating and substituting, the following equation was obtained:
\[ c_{\text{eq2}} = \left( \frac{b}{a} \right)^2 c_2 \] \hspace{1cm} (A.16)

Combining dampers is essentially the same as combining springs. If the force passing through each damper is the same but displacements and velocities are different, then the dampers are in series. When the force passing through each damper is different but the displacements and velocities are the same, then the dampers are in parallel. The effective damper equation for dampers in series and in parallel are the same as that of springs in series and in parallel, respectively. As for the above example, the force passing through each damper is different but their displacements and velocities are the same. Therefore, the effective damping coefficient would be:
\[ c_{\text{eff}} = c_1 + c_{\text{eq2}} = c_1 + \left( \frac{b}{a} \right)^2 c_2 \] \hspace{1cm} (A.17)

The equivalent system is shown below.

\[ \text{Figure A.3 - Equivalent System of the Lever and Gear Ratio Sample} \]

The equivalent model would maintain the dynamic properties of the original without the complexity of the equation of motion. With the knowledge of dynamic
modeling, it would be possible to understand the research done previously, whether it was the simple SDOF or complex SDOF model.
Appendix B: Simulink vs. DYNACAM comparison

Creation of a 2-mass SDOF model

To ensure that the user is capable of modeling cam-follower system in Simulink, the following simple two-mass SDOF model was created.

![Diagram of a 2-mass SDOF model](image)

**Figure B.1 - Simple Industrial 2-Mass 1-DOF Model (Norton et al. 2002)**

The above model has the following equations of motion:

\[
\begin{align*}
\dot{m}_1 \ddot{x}_1 &= -k_{01} x_1 - c_{01} \dot{x}_1 + k_{12} (x_2 - x_1) + c_{12} (\dot{x}_2 - \dot{x}_1) + F_c \quad (B.18) \\
\dot{m}_2 \ddot{x}_2 &= -k_{12} (x_2 - x_1) - c_{12} (\dot{x}_2 - \dot{x}_1) \quad (B.2)
\end{align*}
\]

It is assumed that mass 1 never separates from the cam, which means \(x_1 = s\). To solve the above equation, it is necessary that the input, \(s\), is known. The input function of the above model is an example taken from DYNACAM, which is a double-dwell translating cam-follower system that consists of four segments, as shown in the screenshot of **DYNACAM Input Screen**.
The previous equations were simplified to reduce the modeling complexity in Simulink. The simplified equations which were replicated in Simulink are shown here:

\[ \ddot{x}_2 = 2\zeta_2\omega_2x_1 + \omega_2^2x_1 - 2\zeta_2\omega_2\dot{x}_1 - \omega_2^2x_1 \]  
\[ F_z = m_1\ddot{x}_1 + (c_2 + c_1)\dot{x}_1 + (k_2 + k_1)x_1 - c_2\dot{x}_1 - k_2x_1 \]

By calculating the right hand side of equation 4.3, simulated acceleration was obtained. Integrating the top equation with respect to time gives the simulated velocity and by integrating the velocity, a simulated position was acquired. The contact force was easily calculated by computing the right hand side of equation B.4. This calculation would give an approximate force experienced by the follower at each time step.

In this particular case, the goal was to determine whether Simulink is capable of modeling a cam-follower system. Therefore, the physical parameters were not calculated but arbitrarily chosen. When the final dynamic model of the system is created, these physical properties will be calculated through CAD software to obtain the most accurate values.

**Simulink: 2-Mass SDOF Model**

Having the inputs, the parameters, and the necessary equations, it was possible to create a dynamic model in Simulink. The dynamic model created in Simulink to simulate position, velocity, and acceleration of mass two as well as the contact force is shown in Figure B..
The two equations were modeled as subsystems. In Figure B., the subsystem on the left represents the calculation of simulated position, velocity, and acceleration of mass two, while the subsystem on the right represents the contact force simulation. These subsystems were introduced into the modeling process to minimize confusion as well as maintain the organization needed for multiple variables equations. The blocks shown in green, dark blue, light blue, and red are the inputs of the cam, the masses, stiffness constants, and critical damping coefficients, respectively. Once the calculation of the equation of motion for mass two was completed, the results were transferred to the other subsystem to determine the contact force. In order to see how Simulink represents the equations, the subsystems must be explored. Figure B. is the representation of the equation of motion of mass two.
Because the units of the inputs obtained from DYNACAM were in millimeters, they were multiplied by 1000 to convert to meters. The masses were already in the appropriate unit, kilogram, which can be used directly. The subsystems seen in this figure calculates the damping coefficients, $c_1$ and $c_2$, which are simply:

$$c_1 = 2m_1\zeta \sqrt{ \frac{k_1}{m_1} } \quad c_2 = 2m_2\zeta \sqrt{ \frac{k_2}{m_2} }$$  \hspace{1cm} (B.5)$$

The solutions of the equation and its integrals were plotted in the pink blocks, which are the acceleration, velocity, and displacement. As mentioned previously, the results were transferred to the calculation of the contact force in the other subsystem, as shown in Figure B..
After the contact force calculation was completed, the simulated results were exported into Matlab data files Output Displacement, Output Velocity, and Output Acceleration, shown in the upper right-hand side of Figure B.. These files were converted into a text file, which made it possible to compare them to the simulated results obtained from DYNACAM.

**Validation of Simulink for non-impact model with DYNACAM**

To ensure the validity of the dynamic model created in Simulink, the results obtained must be compared to a reliable source. Depending on the result of the comparison, it would be possible to determine the validity of this Simulink model. Therefore, the model created in the previous section will be compared to DYNACAM’s result.
The common parameters for DYNACAM and Simulink are:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass 1</td>
<td>4 kg</td>
<td></td>
</tr>
<tr>
<td>Mass 2</td>
<td>4 kg</td>
<td></td>
</tr>
<tr>
<td>Stiffness Constant 1</td>
<td>10,000</td>
<td>N/m</td>
</tr>
<tr>
<td>Stiffness Constant 2</td>
<td>600,000</td>
<td>N/m</td>
</tr>
<tr>
<td>Critical Damping 1</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>Critical Damping 2</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>Preload</td>
<td>100 N</td>
<td></td>
</tr>
</tbody>
</table>

The common physical properties, which were arbitrarily chosen, were inserted into DYNACAM’s vibration simulation and Simulink model. Once the calculation was completed, the data was exported into a text file which could be opened in spreadsheet software for comparison purposes. The simulated data obtained were inserted into Excel and are shown in Figure B. through Figure B..

![Simulated Displacement Comparison](image)

**Figure B.6 - DYNACAM vs. Simulink Displacement Comparison**

Figure B. shows the displacement versus cam angle comparison of the simulated data obtained from DYNACAM and Simulink, which are shown in blue and pink, respectively. Unfortunately, the displacement difference is apparent during the upper
dwell, cam angle 90 to 180 degrees. This was due to the significant figure cut-off on Simulink’s part, but this was not determined to be important because the objective of this comparison is to observe other characteristics beside the displacement. Therefore, velocity comparison was completed and is shown in Figure B.

![Simulated Velocity Comparison](image)

**Figure B.7 - DYNACAM vs. Simulink Velocity Comparison**

The velocity versus cam angle comparison shows very little difference between the Simulink model and DYNACAM model. The differences can be seen during the positive and negative peaks at 45 degrees and 225 degrees, respectively. Even at these locations, the differences were determined to be minimal and ignored. While the displacement and velocity may show limited differences, the most sensitive component would be the acceleration simulated result, which is shown in Figure B.
The acceleration versus cam angle comparison once again shows almost no difference between the DYNACAM and Simulink results. The maximum and minimum values of accelerations are practically the same and the frequencies of these two data match very well. From the above comparisons of the displacement, velocity, and acceleration between DYNACAM and Simulink, it would be reasonable to assume that Simulink’s result matches DYNACAM’s result very well. From the above comparisons, it was determined that Simulink may be used to create dynamic model for cam-follower system.
Appendix C: Impact Force Determination

Calculations of the Impact Parameters

Prior to creating a new schematic lumped-parameter model, the impact stiffness constant must be obtained. This stiffness is a combination of the hard-stop’s stiffness and the impact screw’s stiffness. Since the stiffness constant of the hard-stop is already known, the stiffness constant of the impact screw must be determined. Because the tip of the impact screw was ground to create a hemisphere, its deformation is non-linear. To obtain the stiffness constant of the impact screw, two stiffness components must be determined: the stiffness of the round surface and the stiffness of the over-hanging length, represented by Linear + Curved member of Figure 3.9, which are obtainable through the following equations.

\[ \delta_{sphere} = 1.55 \left( \frac{F^2}{E^2 D} \right)^{\frac{1}{3}} \]  
\[ \delta_{axial} = \frac{FL}{AE} \]  

With the known parameters shown below,

Young’s Modulus = 30×10^6 psi
Diameter of the Contact Surface = 3 in
Diameter of the Impact Screw = 0.1876 in (minor diameter of the screw)
X-section Area of the Screw = 0.02764 in²
Overhanging Length of the Screw ≈ 0.4 in

The stiffness constant of the impact screw was easily determined. When \( F = 25 \text{ lb} \), the spherical and axial deflection are: \( \delta_{sphere} = 1.55 \left( \frac{25^2}{(30\times10^6)^2} \right)^{\frac{1}{3}} = 9.52\times10^{-5} \text{ in} \) and \( \delta_{axial} = \frac{25 \times 0.4}{0.02764 \times 30 \times 10^6} = 1.21\times10^{-5} \text{ in} \), respectively. Because these stiffness constants are in series, their deflection were combined and the stiffness constant of the impact screw was found to be: \( k_{ss} = \frac{25}{1.07 \times 10^{-4}} = 233,644 \text{ lb/in} \). Combining the stiffness constant
of the impact screw and hard-stop yields the following stiffness constant:

\[ k_{03} = \frac{k_{is} \times k_{hs}}{k_{is} + k_{hs}} = \frac{233,644 \times 60800}{233,644 + 60800} = 48,245 \text{ lb/in}. \]

**Validation of Impact Force Approximation: Ball Drop Experiment**

Prior to performing experiments on the CDTM with impact and over-travel, it was important to obtain the superior impact force approximation method. As mentioned in Chapter 3.3, there are two main methods of approximating impact force, which are the energy and wave methods. Although the wave method gives a more accurate approximation, the complexities exceed the accuracy advantage. The energy method was considered and researched. The two energy methods found to be applicable were, as described in *Mechanical Analysis and Design* (Burr 1982) and *Impact Forces in Mechanisms* (Johnson 1958), deflection and correction factor approach and relative velocity approaches, respectively.

To determine the best impact force approximation, a simple impact experiment was created. This is a striking impact that occurs when a spherical ball strikes a flat surface. The repeatability and multiple impact velocities were the major concerns in this experiment. To obtain a repeatable experiment, the conditions of the impact must be consistent while obtaining multiple impact velocities requires multiple drop heights. With these two conditions in mind, an L-bracket with a hole at the center was designed, manufactured, and attached to the THK LM cart. The end stopper that came with the THK rail was used as a resting place for the THK LM cart. The combinations of these two parts allow a steady and adjustable drop height for multiple data points to be averaged.

Figure C.1 shows the ball drop experiment setup with the stopper in place. As seen in this picture, the two dimensions \( D_1 \) and \( D_2 \) are known. Therefore, the drop height...
is $D_2 - D_1$. From the known relative height, it is possible to obtain the impact velocity from the following equation:

$$V_i^2 = V_o^2 + 2gh$$  \hspace{1cm} (C.3) \nonumber$$

where $V_o$, $V_i$, $g$, and $h$ are the impact velocity, initial velocity, gravitational acceleration, and relative height, respectively. Because the initial velocity is zero, the final equation for impact velocity is:

$$V_i = \sqrt{2gh}$$  \hspace{1cm} (C.4) \nonumber$$

The impact velocity is needed in order to obtain the impact force approximation through common velocity approach while only the relative height is needed for the deflection and correction factor approach.

The heights at which the ball was dropped from were 1 in, 1.25 in, 1.75 in, and 2.25 in. Only the first impact data was saved in the oscilloscope by setting a trigger delay. Multiple drops were performed to obtain averaged data for these drop heights. By connecting the force transducer, Dytran 1050V3, to an HP 54503A 500 MHz Oscilloscope, the following data were obtained:

<table>
<thead>
<tr>
<th>Height (in)</th>
<th>Voltage (V)</th>
<th>Force (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.00</td>
<td>2.80</td>
<td>51.95</td>
</tr>
<tr>
<td>1.25</td>
<td>3.00</td>
<td>55.66</td>
</tr>
<tr>
<td>1.75</td>
<td>3.45</td>
<td>64.01</td>
</tr>
<tr>
<td>2.25</td>
<td>4.00</td>
<td>74.21</td>
</tr>
</tbody>
</table>

Table 5 - Ball Drop Experiment Output

Table 5 shows the different heights at which the ball was dropped and their respective voltage and force outputs. Even though the default output unit of the oscilloscope is in Volts, it was easily converted into pound force with the known sensitivity of 53.9 mV per pound force. These data will be compared to the theoretical data computed through deflection and correction factor and common velocity approaches.
The deflection and correction factor was the first method that was derived for this comparison. The original form is not suitable for this particular experiment. Therefore a new equation must be derived from the known properties. The original deflection and correction factor approach has the following form:

\[
F_{i.o.\text{def}} = \left(1 + \frac{2h}{\delta_{st}} \eta\right) W' \tag{C.5}
\]

As the name applies, the equation contains both deflection factor, \(\delta_{st}\), and correction factor, \(\eta\). For this experiment, the correction factor was substituted by one because of the large driven mass to driving mass. A new equation was formed from the mentioned simplification and is shown below.

\[
F_{i.\text{def}} = \left(1 + \frac{2h}{\delta_{st}}\right) W' \tag{C.6}
\]

To calculate the above equation, three values must be determined: the drop height, \(h\), static deflection due to the weight of driving mass, \(\delta_{st}\), and the weight of the driving mass, \(W'\). The drop height and the weight of the driving mass could easily be obtained and calculated whereas the static deflection required more calculations. The heights were easily obtained from the experimental heights. The weight of the driving mass is simply:

\[
W' = \frac{4}{3} \pi \left(\frac{\text{Dia}}{2}\right)^3 \cdot \rho \cdot g \tag{C.7}
\]

where \(\text{Dia}, \rho,\) and \(g\) are the diameter of the spherical ball, the mass density of the ball, and the gravitational constant. The static deflection was obtained by calculation and experimentation. The experimentation was obtaining the stiffness constant of the hard-stop when fully assembled to the stanchion and placing a dial indicator to obtain the displacement due to the force applied. A stiffness constant of the hard-stop, \(k_{hs}\), was obtained and used in the calculation of the deflection factor. With the following values \(\text{Dia} = 0.375\ \text{in}, \rho = 0.0007349\ \text{lb/in}^3, g = 386.4\ \text{in/s}^2, k_{hs} = 60800\ \text{lb/in},\) the two constants obtained are \(W' = 7.841 \times 10^{-3}\ \text{lb}\) and \(\delta_{st} = 1.289 \times 10^{-7}\ \text{in}\) and the final deflection and correction factor impact force equation for this particular experiment is:
Different drop heights, \( h \), were inserted into the above equation and will be compared to the experimental as well as common velocity approach.

The common velocity approach was much simpler to employ because the equation for the case where a spherical ball strikes a flat object had already been presented in Ray C. Johnson’s paper. An altered form of the common velocity equation is shown below.

\[
F_{i \_cv} = C \left( \frac{1}{2} \right) \left( \frac{5}{4} m(V_i)^2 \right) \left( \frac{1}{2} \right) \tag{C.9}
\]

where \( C \), \( m \), and \( V_i \) are the proportionality constant, driving mass, and the impact velocity. The proportionality constant could easily be calculated using the following equation for a spherical ball striking a flat object.

\[
C = 0.518 \cdot E \cdot \sqrt{\text{Dia}} \tag{C.10}
\]

where \( E \) and \( \text{Dia} \) are the modulus of elasticity and the diameter of the spherical ball which are \( 2.9 \times 10^7 \text{ lb/ in}^2 \) and 0.375 in, respectively. Knowing these two values, the proportionality constant was calculated as \( 9.2 \times 10^6 \).

The mass of the spherical ball is simply \( m_2 = \frac{4}{3} \pi \left( \frac{\text{Dia}}{2} \right)^3 \rho \) or 2.029 \( \times 10^{-5} \) slugs while the impact velocity could be derived from the equation shown earlier in this section, \( V_i = \sqrt{2gh} \). Substituting the known values into the original equation, \( F_{i \_cv} \), the following equation was obtained:

\[
F_{i \_cv} = \left( 9.2 \times 10^6 \right)^{\frac{1}{2}} \left( \frac{5}{2} \right) \left( 2.029 \times 10^{-5} \right) \left( 385.4 \right) \times h \left( \frac{1}{2} \right) \Rightarrow F_{i \_cv} = 57.66 \times h^{\frac{3}{2}} \tag{C.11}
\]

With both deflection and correction factor and common velocity impact force approximation equations in the simplest form, the drop heights were inserted into the two equations and Table 6 was obtained.
Table 6 compares the calculated and experimental impact forces obtained through experiment, the deflection and correction factor energy method, and the common velocity approach. Table 6 and Figure C. show that deflection and correction factor method underestimates the actual impact force whereas the common velocity approach predicts higher forces for every drop height.
Figure C.3 - Ball Drop Experiment Percent Error Comparison

Figure C. shows the percent error relative to the experiment between the deflection and correction factor and the common velocity approach. As seen in the Figure C., the deflection and correction factor method was between 36% and 40% lower, but the common velocity overestimated the impact forces by 10% to 26%. This overestimation automatically gives the approximation a minimum factor of safety of 1.1. From the above result, it would seem appropriate to use the common velocity to approximate impact forces for the modeling of cam-follower system with impact loading.
Arthur H. Burr – Deflection and Correction Factor Approach

\[ k = \frac{\text{Weight}}{\text{Displacement}} \]
\[ W = \frac{4}{3} \cdot \pi \cdot \left( \frac{\text{Dia}}{2} \right)^3 \cdot \rho \cdot g \cdot \delta_{\text{eff}} = \frac{W}{k} \]
\[ k = 1.064771159 \times 10^9 \]
\[ W = 0.01111238171 \pi \]
\[ \delta_{\text{eff}} = 1.044579543 \times 10^{-5} \pi \]

\[ \sigma_{\text{valf}}(P[1]) = W \cdot \left( 1 + \sqrt{1 + \frac{2 \cdot \delta_{\text{eff}}[1]}{\delta_{\text{eff}}}} \right) \]
\[ \sigma_{\text{valf}}(P[2]) = W \cdot \left( 1 + \sqrt{1 + \frac{2 \cdot \delta_{\text{eff}}[2]}{\delta_{\text{eff}}}} \right) \]
\[ \sigma_{\text{valf}}(P[3]) = W \cdot \left( 1 + \sqrt{1 + \frac{2 \cdot \delta_{\text{eff}}[3]}{\delta_{\text{eff}}}} \right) \]
\[ \sigma_{\text{valf}}(P[4]) = W \cdot \left( 1 + \sqrt{1 + \frac{2 \cdot \delta_{\text{eff}}[4]}{\delta_{\text{eff}}}} \right) \]

\[ F_1 = 137.5131441 \]
\[ F_2 = 153.7406437 \]
\[ F_3 = 181.9015962 \]
\[ F_4 = 206.2522414 \]
ANSI Units

\[ k = \frac{\text{Weight}}{\text{Displacement}} \]

\[ N = \frac{4}{3} \cdot a \cdot \left( \frac{D a}{2} \right)^{3/a} \cdot g \cdot \delta \cdot \delta' \]

\[ \delta' = 4.125 \times 10^{-3} \cdot x \]

\[ \text{Displacement} = 0.0025 \]

\[ \text{Weight} = 152 \]

\[ \text{Displacement} = 0.0025 \]

\[ \delta = 0.0003 \]

\[ \delta' = 0.0003 \]

\[ x = 4.035 \times 10^{-3} \cdot x \]

\[ \delta' = 4.135 \times 10^{-3} \cdot x \]

\[ \text{Displacement} = 0.0025 \]

\[ \text{Weight} = 152 \]

\[ \text{Displacement} = 0.0025 \]

\[ \delta = 0.0003 \]

\[ \delta' = 0.0003 \]

\[ x = 4.035 \times 10^{-3} \cdot x \]

\[ \delta' = 4.135 \times 10^{-3} \cdot x \]
Ray C. Johnson – Common Velocity Approach

\section*{SI Units}

\begin{verbatim}
> restart;
> \theta = 2\pi / 11; \Delta x = \frac{0.375 \cdot 25 \cdot 4}{1000}; \rho = 7872; g = 9.81; h_1 = 0.0254; h_2 = 0.04445; h_3 = 0.05715; h_4 = 0.111

\theta = 2 \cdot \frac{\pi}{11}
\Delta x = 0.000625 \cdot \text{mm}
\rho = 7872
g = 9.81
h_1 = 0.0254
h_2 = 0.04445
h_3 = 0.05715
h_4 = 0.111

\begin{equation}
\nu = \frac{\pi \cdot \Delta x}{6} \cdot m \cdot [z] = \nu \cdot C = 0.518 \cdot \nu \cdot \text{sqrt}(\Delta x);
\end{equation}

\text{m}_1 = 0.00113379991 \text{ m}
C = 1.011695663 \times 10^{10}
\begin{equation}
\nu = \text{sqrt}(2 \cdot g \cdot h_1), \nu_2 = \text{sqrt}(2 \cdot g \cdot h_2), \nu_3 = \text{sqrt}(2 \cdot g \cdot h_3), \nu_4 = \text{sqrt}(2 \cdot g \cdot h_4) \text{.}
\end{equation}

\text{v}_1 = 0.789395743\text{ m/s}
\text{v}_2 = 0.789262318\text{ m/s}
\text{v}_3 = 0.533347063\text{ m/s}
\text{v}_4 = 1.089065311\text{ m/s}

\begin{equation}
\text{evalf}[P[1]] = C^2 \cdot \left( \frac{5 \cdot m_2}{4} \cdot (\nu_1)^2 \right)^{\frac{3}{2}}, \text{evalf}[P[2]] = C^2 \cdot \left( \frac{5 \cdot m_2}{4} \cdot (\nu_2)^2 \right)^{\frac{3}{2}}, \text{evalf}[P[3]] = C^2 \cdot \left( \frac{5 \cdot m_2}{4} \cdot (\nu_3)^2 \right)^{\frac{3}{2}}, \text{evalf}[P[4]] = C^2 \cdot \left( \frac{5 \cdot m_2}{4} \cdot (\nu_4)^2 \right)^{\frac{3}{2}}
\end{equation}

\begin{itemize}
\item \text{P}_1 = 256.9469754
\item \text{P}_2 = 293.5862711
\item \text{P}_3 = 325.3630158
\item \text{P}_4 = 417.7954498
\end{itemize}
\end{verbatim}
\textbf{ANSI UNITS}

\begin{itemize}
\item \texttt{restart;}
\item \texttt{B = 29.6, Da = 0.375, \rho = \frac{0.264}{366.4}, g = 366.4, h[1] = 1, h[2] = 1.25, h[3] = 1.75, h[4] = 2.25,}
\item \texttt{E = 2 \times 10^7}
\item \texttt{Da = 0.295}
\item \texttt{g = 0.000733 \times 0.000166}
\item \texttt{g = 366.4}
\item \texttt{h[1] = 1}
\item \texttt{h[2] = 1.25}
\item \texttt{h[3] = 1.75}
\item \texttt{h[4] = 2.25}
\item \texttt{V = \frac{\pi \cdot Da \cdot E \cdot h[2]}{6}}
\item \texttt{V; \rho; C = \frac{511 \cdot E \cdot sqrt(Da \cdot h[2])}{(2 \cdot g \cdot h[2])}}
\end{itemize}

\begin{align}
\texttt{m_2} &= \frac{0.00000045906333 \times 10^{-6}}{C} \\
\texttt{C} &= 1.99911792 \times 10^6
\end{align}

\begin{align}
\texttt{v[1]} &= \sqrt{2 \cdot g \cdot h[1]} \quad \texttt{v[2]} = \sqrt{2 \cdot g \cdot h[2]} \\
\texttt{v[3]} &= \sqrt{2 \cdot g \cdot h[3]} \quad \texttt{v[4]} = \sqrt{2 \cdot g \cdot h[4]}
\end{align}

\begin{align}
\texttt{r_1} &= 27.75430577 \\
\texttt{r_2} &= 31.08034854 \\
\texttt{r_3} &= 36.77699130 \\
\texttt{r_4} &= 41.95892015
\end{align}

\begin{align}
\texttt{eval}[F[1]] &= C^2 \cdot \left( \frac{5 \cdot m[2]}{4} \cdot (r[1])^2 \right)^{\frac{3}{2}} \\
\texttt{eval}[F[2]] &= C^2 \cdot \left( \frac{5 \cdot m[2]}{4} \cdot (r[2])^2 \right)^{\frac{3}{2}} \\
\texttt{eval}[F[3]] &= C^2 \cdot \left( \frac{5 \cdot m[2]}{4} \cdot (r[3])^2 \right)^{\frac{3}{2}} \\
\texttt{eval}[F[4]] &= C^2 \cdot \left( \frac{5 \cdot m[2]}{4} \cdot (r[4])^2 \right)^{\frac{3}{2}}
\end{align}

\begin{align}
\texttt{F_1} &= 29.66410283 \\
\texttt{F_2} &= 65.92556023 \\
\texttt{F_3} &= 29.67342381 \\
\texttt{F_4} &= 93.30031047
\end{align}
Appendix D: Lumped Masses Calculation

1. Link Arm (Pivot Side)  
   I_{zz} = 41.2339 \text{ lb}\times\text{in}^2  
   m = 1.4931 \text{ lb}  
   Material:  Beam = 1060 Alloy,  Ring = Aluminum Bronze  
   Distant:  From pivot to cam follower = 6.5 \text{ in}  

2. Link Arm (Connecting Rod Side)  
   I_{zz} = 103.6399 \text{ lb}\times\text{in}^2  
   m = 0.7208 \text{ lb}  
   Material:  Beam = 1060 Alloy,  Ring = Aluminum Bronze  
   Distant:  From Connecting Rod joint to cam follower = 13.5 \text{ in}
<table>
<thead>
<tr>
<th>Component Description</th>
<th>Mass (lb)</th>
<th>Material</th>
</tr>
</thead>
<tbody>
<tr>
<td>3. Connecting Rod (Link Arm Half)</td>
<td>0.5035</td>
<td>1060 Alloy</td>
</tr>
<tr>
<td>4. Connecting Rod (Arm Rocker Half)</td>
<td>0.2372</td>
<td>1060 Alloy</td>
</tr>
<tr>
<td>5. Arm Rocker (Connecting Rod Side)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Izz = 9.2433 lb×in²</td>
<td>0.3357</td>
<td>1060 Alloy, Aluminum Bronze</td>
</tr>
<tr>
<td>Distant: From pivot to Connecting Rod joint = 7.75 in</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6. Arm Rocker (Impact Mechanism Side)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Izz = 0.5213 lb×in²</td>
<td>0.1276</td>
<td>1060 Alloy, Aluminum Bronze</td>
</tr>
<tr>
<td>Distant: From pivot to Connecting Rod joint = 2.375 in</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7. Bottom Rod End</td>
<td>0.1155</td>
<td>Steel</td>
</tr>
<tr>
<td>8. Top Rod End</td>
<td>0.2459</td>
<td>Adaptor &amp; Enclosure Sleeve = Brass</td>
</tr>
<tr>
<td>9. Rod End Block</td>
<td>0.0844</td>
<td></td>
</tr>
<tr>
<td>Material:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rod End Block = 1060 Alloy</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Impact Screw = Galvanized Steel</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SCS M2.6x0.45mm = Steel (Black-Oxide)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SCS #8-32 = 4140 Alloy Steel</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10. THK LM Cart</td>
<td>0.0551</td>
<td></td>
</tr>
</tbody>
</table>
For all rotating links:  \( I_z = m \times r^2 \)

1. \( 41.2339 \text{ lb}\times\text{in}^2 = m \times (13.5 \text{ in})^2 \)  \( \Rightarrow m_1 = 0.2262 \text{ lb} \)
2. \( 103.6399 \text{ lb}\times\text{in}^2 = m \times (13.5 \text{ in})^2 \)  \( \Rightarrow m_2 = 0.5686 \text{ lb} \)
3. \( 9.2433 \text{ lb}\times\text{in}^2 = m \times (7.75 \text{ in})^2 \)  \( \Rightarrow m_5 = 0.1539 \text{ lb} \)
4. \( 0.5213 \text{ lb}\times\text{in}^2 = m \times (2.375 \text{ in})^2 \)  \( \Rightarrow m_6 = 0.0924 \text{ lb} \)

\( \text{mla} = 1 + 2 \) \hspace{1cm} \text{Link Arm (Pivot Side) + Link Arm (Conn Rod Side)}

\( \text{mcr} = 3 + 4 \) \hspace{1cm} \text{Connecting Rod (Lower 1/2) + Connecting Rod (Upper 1/2)}

\( \text{mar} = 5 \) \hspace{1cm} \text{Arm Rocker Right (ConnRod Side)}

\( \text{marl} = 6 + 7 \) \hspace{1cm} \text{Arm Rocker Left (Impact Mechanism Side)}

\( \text{mbre} = 7 \) \hspace{1cm} \text{Bottom Rod End}

\( \text{mcre} = 8 + 9 + 10 \) \hspace{1cm} \text{Top Rod End + Rod End Block + THK LM Cart}
Combined Masses:

$m_1 = m_{la} + m_{cr} = 0.7949 + 0.7407 = 1.5356 \text{ lb}$

$m_2 = m_{arr} = 0.1539 \text{ lb}$

$m_3 = m_{arl} + m_{bre} = 0.0924 + 0.1155 = 0.2079 \text{ lb}$

$m_4 = m_{tre} = 0.3854 \text{ lb}$

Combining mass 2 and 3 together and translating it to the intermediate mass’s location:

$M_2 = m_4 \times \frac{r_3}{r_4^2} \Rightarrow \quad M_2 = 0.2079 + \frac{0.1539 \times 7.75^2}{2.375^2} \Rightarrow \quad M_2 = 1.6388 \text{ lb}$

Translating mass 1 to the end effector location:

$M_1 = m_1 \times \frac{r_3}{r_4^2} \Rightarrow \quad M_1 = \frac{1.5356 \times 7.75^2}{2.375^2} \Rightarrow \quad M_1 = 16.35 \text{ lb}$
Masses at the intermediate mass:
M1 = 16.351 lb
M2 = 1.6388 lb
M3 = 0.3854 lb
Appendix E: Stiffness Constants Calculation

Closure Spring

Using the following equation, the designer was able to calculate the spring rate of the spring used in the machine.

$$ k = \frac{d^4 G}{8 D^3 N_a} $$

Equation 1 : Helical Extension Spring Rate

where

- $k$ = spring rate (lb/in)
- $d$ = the wire diameter (in)
- $G$ = Modulus of Rigidity (psi)
- $D$ = mean coil diameter (in)
- $N_a$ = Number of active coils – rounded to 1/4 coil

After measuring the spring in the lab the designer found the above variables to be:

- $d = 0.136$ (in)
- $G = 11.7 \times 10^6$ (psi)
- $D = 0.825$ (in)
- $N_a = 34.75$

Substituting the above values into Equation 1, the designer found that the spring rate used in the cam machine is:

$$ Spring \_ \_ \_ Rate = 25.64 \frac{lb}{in} $$

As for the preload of the spring, the designer found that the rested spring length is approximately 4.375 inches and extended spring at the low dwell is 6.5 inches. Also, the manufacturer preload on the spring was found to be 14 lbf according to the Design of Machinery book. Therefore, the preload was calculated as follow:

$$ Preload = Spring \_ \_ \_ rate \times (Extended \_ \_ \_ Length - Rested \_ \_ \_ Length) + Manufacturer \_ \_ \_ Preload $$

Preload = 25.64 lbf/in $\times (6.5 \_ \_ \_ in - 4.375 \_ \_ \_ in) + 14 \_ \_ \_ lbf = 68.485 \_ \_ \_ lbf $
Link Arm

Summing the force in the vertical direction:
\[ \sum F_y = 0 = F + R_1 - R_2 \]
\[ F = R_2 - R_1 \]

Summing the moment about the pivot (\( R_1 \)):
\[ \sum M = 0 = F \times b - R_2 \times a \]
\[ R_2 = \frac{F \times b}{a} \]

Substituting \( R_2 \) into \( F = R_2 - R_1 \) in order to determine \( R_1 \) gives:
\[ R_1 = \frac{F \times b}{a} - F \]
\[ R_1 = \frac{Fb - Fa}{a} \]

Link arm is rotating about the pin connection at reaction 1 (\( R_1 \)). The cam-follower is in contact with the cam at reaction 2 (\( R_2 \)) and pulled at the right by force \( F \). With this information it would be possible to create the loading, shear, and moment functions.

Loading function

\[ \text{Preload} = 68.485 \text{lb} \]
\[ q = R_1 \langle x - 0 \rangle^{-1} - R_2 \langle x - a \rangle^{-1} + F \langle x - b \rangle^{-1} \]

Shear function
\[ V = \int q \, dx = R_1 \langle x - 0 \rangle^0 - R_2 \langle x - a \rangle^0 + F \langle x - b \rangle^0 + C_1 \]

Moment function
\[ M = \int V \, dx = R_1 \langle x - 0 \rangle^1 - R_2 \langle x - a \rangle^1 + F \langle x - b \rangle^1 + C_1 x + C_2 \]

Slope function
\[ \theta = \int \frac{M}{EI} \, dx = \frac{1}{EI} \left( \frac{R_1}{2} \langle x - 0 \rangle^2 - \frac{R_2}{2} \langle x - a \rangle^2 + \frac{F}{2} \langle x - b \rangle^2 + \frac{C_1}{2} x^2 + C_2 x + C_3 \right) \]

Deflection function
\[ y = \int \frac{\theta}{EI} \, dx = \frac{1}{EI} \left( \frac{R_1}{6} \langle x - 0 \rangle^3 - \frac{R_2}{6} \langle x - a \rangle^3 + \frac{F}{6} \langle x - b \rangle^3 + \frac{C_1}{6} x^3 + \frac{C_2}{2} x^2 + C_3 x + C_4 \right) \]

C_1 and C_2 are zero because they are included in the loading function. Deflections at the supports (R_1 and R_2) are also zero which resulted in the conditions of \( x = 0, y = 0 \) and \( x = a, y = 0 \).

\[ q = R_1 \langle x - 0 \rangle^{-1} - R_2 \langle x - a \rangle^{-1} + F \langle x - b \rangle^{-1} \]
\[ V = R_1 \langle x - 0 \rangle^0 - R_2 \langle x - a \rangle^0 + F \langle x - b \rangle^0 \]
\[ M = R_1 \langle x - 0 \rangle^1 - R_2 \langle x - a \rangle^1 + F \langle x - b \rangle^1 \]
\[ \theta = \frac{1}{EI} \left( \frac{R_1}{2} \langle x - 0 \rangle^2 - \frac{R_2}{2} \langle x - a \rangle^2 + \frac{F}{2} \langle x - b \rangle^2 + C_3 \right) \]
\[ y = \frac{1}{EI} \left( \frac{R_1}{6} \langle x - 0 \rangle^3 - \frac{R_2}{6} \langle x - a \rangle^3 + \frac{F}{6} \langle x - b \rangle^3 + C_3 x + C_4 \right) \]

When \( x = 0 \Rightarrow y = 0 \quad *b > a\)
\[ y = 0 = \frac{1}{EI} \left( \frac{R_1}{6} \langle 0 - 0 \rangle^3 - \frac{R_2}{6} \langle 0 - a \rangle^3 + \frac{F}{6} \langle 0 - b \rangle^3 + C_3 \times 0 + C_4 \right) \]
\[ C_4 = -\frac{R_1}{6} \langle 0 - 0 \rangle^3 + \frac{R_2}{6} \langle 0 - a \rangle^3 - \frac{F}{6} \langle 0 - b \rangle^3 - C_3 (0) \]
\[ C_4 = -\frac{R_1}{6} (0) + \frac{R_2}{6} (0) - \frac{F}{6} (0) - C_3 (0) = 0 \]

When \( x = a \Rightarrow y = 0 \quad *b > a\)
\[ y = 0 = \frac{1}{EI} \left( \frac{R_1}{6} \langle a - 0 \rangle^3 - \frac{R_2}{6} \langle a - a \rangle^3 + \frac{F}{6} \langle a - b \rangle^3 + C_3 \times a + 0 \right) \]
\[ C_3 = -\frac{R_1}{6a} \langle a - 0 \rangle^3 + \frac{R_2}{6a} \langle a - a \rangle^3 - \frac{F}{6a} \langle a - b \rangle^3 \]
because \( b \) is greater than \( a \), the singularity function in the third term equals to 0
\[
C_3 = \frac{R_a}{6a} (a)^3 + \frac{R_0}{6a} (0)^3 - \frac{F}{6a} (0)^3 \rightarrow C_3 = -\frac{R_a}{6a} (a)^3
\]

making the appropriate substitution for \( R_1 = \frac{Fb - Fa}{a} \) gives:
\[
C_3 = -\frac{Fb - Fa}{6a^2} (a)^3 \rightarrow C_3 = -\frac{Fb - Fa}{6} (a) \rightarrow C_3 = -\frac{b - a}{6} (aF)
\]

The newest equations for loading, shear, moment, slope, and deflection functions are:
\[
q = R_1 (x - 0)^{-1} - R_2 (x - a)^{-1} + F (x - b)^{-1}
\]
\[
V = R_1 (x - 0)^0 - R_2 (x - a)^0 + F (x - b)^0
\]
\[
M = R_1 (x - 0)^1 - R_2 (x - a)^1 + F (x - b)^1
\]
\[
\theta = \frac{1}{EI} \left( \frac{R_1}{2} (x - 0)^2 - \frac{R_2}{2} (x - a)^2 + \frac{F}{2} (x - b)^2 - \frac{b - a}{6} (aF) \right)
\]
\[
y = \frac{1}{EI} \left( \frac{R_1}{6} (x - 0)^3 - \frac{R_2}{6} (x - a)^3 + \frac{F}{6} (x - b)^3 - \frac{b - a}{6} (aF) x \right)
\]

Substituting the values for \( F = 100 lb \), \( a = 6.5 in \), \( b = 13.5 in \), \( E = 10,007,603 \) psi (Aluminum):
\[
R_1 = \frac{Fb - Fa}{a} = \frac{100(13.5) - 100(6.5)}{6.5} = 107.69
\]
\[
R_2 = \frac{F \times b}{a} = \frac{100(13.5)}{6.5} = 207.69
\]
\[
I = \frac{bh^3}{12} = \frac{1 \times 1 \times 1.5^3}{12} = \frac{3.375}{12} = 0.28125 \text{ in}^4
\]
\[
y = \frac{1}{10,007,603 \times 0.28125} \left( \frac{107.69}{6} (x)^3 - \frac{207.69}{6} (x - 6.5)^3 + \frac{100}{6} (x - 13.5)^3 \right)
\]
\[
y = \frac{1}{2,814,638} \left( 17.95(x)^3 - 34.62(x - 6.5)^3 + 16.67(x - 13.5)^3 - 758.33x \right)
\]

When \( x = 13.5 \) the deflection is:
\[
y = \frac{1}{2,814,638} \left( 17.95(13.5)^3 - 34.62(13.5 - 6.5)^3 + 16.67(13.5 - 13.5)^3 - 758.33(13.5) \right)
\]
\[
y = \frac{1}{2,814,638} \left( 17.95(13.5)^3 - 34.62(7)^3 + 16.67(0)^3 - 758.33(13.5) \right)
\]
\[
y = \frac{1}{2,814,638} \left( 44,163.73 - 11,874.66 - 10,237.46 \right) = \frac{22,051.61}{2,814,638} = 7.835 \times 10^{-3}
\]
Deflection @ $x = 13.5\text{in}$ with a 100 lb force applied is $7.835 \times 10^{-3} \text{in}$

COSMOSWorks calculated the deflection of the Link Arm to be in the range of $7.842 \times 10^{-3}$ to $8.618 \times 10^{-3}$. Considered the assumption made in the singularity function, constant area moment of inertia, the difference is relatively small

$$\left(\frac{8.618 \times 10^{-3} + 7.842 \times 10^{-3}}{2}\right) = 8.23 \times 10^{-3}$$

$$\Rightarrow \frac{8.23 \times 10^{-3} - 7.842 \times 10^{-3}}{7.842 \times 10^{-3}} \times 100 = 4.94\%.$$  

In conclusion, the results are as follows:
COSMOS/Works = $8.23 \times 10^{-3}$ in
Theoretical (Singularity) = $7.842 \times 10^{-3}$
Percent Different = 4.94%

$$K_{\text{link}\_\text{arm}} = \frac{100}{8.23 \times 10^{-3}} = \frac{12,150 \text{ lb}}{\text{in}}$$
Since the boundary conditions of the arm rocker right is the same as that of link arm, it is possible to utilize the same equations for loading, shear, moment, slope, and deflection functions, which are:

\[ q = R_1(x-a)^{-1} - R_2(x-a)^{-1} + F(x-b)^{-1} \]
\[ V = R_1(x-a)^0 - R_2(x-a)^0 + F(x-b)^0 \]
\[ M = R_1(x-a)^1 - R_2(x-a)^1 + F(x-b)^1 \]
\[ \theta = \frac{1}{EI} \left( \frac{R_1}{2}(x-a)^2 - \frac{R_2}{2}(x-a)^2 + \frac{F}{2}(x-b)^2 - \frac{b-a}{6} (aF) \right) \]
\[ y = \frac{1}{EI} \left( \frac{R_1}{6}(x-a)^3 - \frac{R_2}{6}(x-a)^3 + \frac{F}{6}(x-b)^3 - \frac{b-a}{6} (aF)x \right) \]

Substituting the values for \( F=100\text{lb}, a=2.375\text{in}, b=10.125\text{in}, E=10,007,603 \text{ psi} \) (Aluminum):

\[ R_1 = \frac{Fb-Fa}{a} = \frac{100(10.125)-100(2.375)}{2.375} = 326.32 \]
\[ R_2 = \frac{F \times b}{a} = \frac{100(10.125)}{2.375} = 426.32 \]
\[ I = \frac{1}{12}bh^3 = \frac{1}{12} \times 0.5 \times 0.75^3 = \frac{0.2109}{12} = 0.01758 \text{ in}^4 \]
\[ y = \frac{1}{10,007,603 \times 0.01758} \left( \frac{326.32}{6}(x)^3 - \frac{426.32}{6}(x-2.375)^3 + \frac{100}{6}(x-10.125)^3 \right) \]
\[ = \frac{1}{175,915} \left( 54.39(x)^3 - 71.05(x-2.375)^3 + 16.67(x-10.125)^3 - 306.77x \right) \]

When \( x = 10.125 \) the deflection is:

\[ y = \frac{1}{175,915} \left( 54.39(10.125)^3 - 71.05(10.125-2.375)^3 + 16.67(10.125-10.125)^3 - 306.77(10.125) \right) \]
\[ y = \frac{1}{175,915} \left( 54.39(10.125)^3 - 71.05(7.75)^3 + 16.67(0)^3 - 306.77(10.125) \right) \]

\[ y = \frac{1}{175,915} \left( 56,455 - 33,072 - 3106 \right) = \frac{20,277}{175,915} = 0.1153 \]

Deflection @ \( x = 10.125 \text{in} \) with a 100 lb force applied is 0.1153 in

COSMOSWorks calculated the deflection of the Arm Rocker (right) to be in the range of 0.1114 to 0.1238. Considered the assumption made in the singularity function, constant area moment of inertia, the difference is relatively small

\[ \left( \frac{0.1238 + 0.1114}{2} = 0.1176 \right) \Rightarrow \frac{0.1176 - 0.1153}{0.1153} \times 100 = 1.99\% . \]

In conclusion, the results are as follows:
COSMOS/Works = 0.1176 in
Theoretical (Singularity) = 0.1153 in
Percent Different = 1.99\% (smaller than FEA)

\[ K_{\text{arm \_ rocker \_ r}} = \frac{100}{0.1176} = 850.34 \frac{lb}{in} \]
Arm Rocker Left

Since the boundary conditions of the arm rocker left is the same as that of link arm, it is possible to utilize the same equations for loading, shear, moment, slope, and deflection functions, which are:

\[
q = R_1(x-0)^{-1} - R_2(x-a)^{-1} + F(x-b)^{-1}
\]
\[
V = R_1(x-0)^0 - R_2(x-a)^0 + F(x-b)^0
\]
\[
M = R_1(x-0)^1 - R_2(x-a)^1 + F(x-b)^1
\]
\[
\theta = \frac{1}{EI} \left( \frac{R_1}{2} (x-0)^2 - \frac{R_2}{2} (x-a)^2 + \frac{F}{2} (x-b)^2 - \frac{b-a}{6} (aF) \right)
\]
\[
y = \frac{1}{EI} \left( \frac{R_1}{6} (x-0)^3 - \frac{R_2}{6} (x-a)^3 + \frac{F}{6} (x-b)^3 - \frac{b-a}{6} (aF)x \right)
\]

Substituting the values for \( F=100\text{lb}, \ a=7.75\text{in}, \ b=10.125\text{in}, \ E=10,007,603 \text{ psi (Aluminum)}: \)

\[
R_1 = \frac{Fb-Fa}{a} = \frac{100(10.125)-100(7.75)}{7.75} = 30.65
\]
\[
R_2 = \frac{F \times b}{a} = \frac{100(10.125)}{7.75} = 130.65
\]
\[
I = \frac{1}{12} bh^3 = \frac{1\times 0.5 \times 0.75^3}{12} = \frac{0.2109}{12} = 0.01758 \text{ in}^4
\]
\[
y = \frac{1}{10,007,603 \times 0.01758} \left( \frac{30.65}{6} (x)^3 - \frac{130.65}{6} (x-7.75)^3 + \frac{100}{6} (x-10.125)^3 \right) - \frac{10.125-7.75}{6} (7.75 \times 100)x
\]

\[
y = \frac{1}{175,915} \left( 5.11(x)^3 - 21.77(x-2.375)^3 + 16.67(x-10.125)^3 - 306.77x \right)
\]

When \( x = 10.125 \) the deflection is:

\[
y = \frac{1}{175,915} \left( 5.11(10.125)^3 - 21.77(10.125-7.75)^3 + 16.67(10.125-10.125)^3 - 306.77(10.125) \right)
\]

\[
y = \frac{1}{175,915} \left( 5.11(10.125)^3 - 21.77(2.375)^3 + 16.67(0)^3 - 306.77(10.125) \right)
\]
\[
y = \frac{1}{175,915} (5,304 - 291.64 - 3106) = \frac{1,906}{175,915} = 0.01083
\]

Deflection @ \( x = 10.125 \) in with a 100 lb force applied is 0.01083 in

COSMOSWorks calculated the deflection of the Arm Rocker (Left) to be in the range of 0.01034 to 0.01175. Considered the assumption made in the singularity function, constant area moment of inertia, the difference is relatively small

\[
\left( \frac{0.01175 + 0.01034}{2} = 0.01154 \right) \Rightarrow \frac{0.01154 - 0.01083}{0.01083} \times 100 = 6.60\%.
\]

In conclusion, the results are as follows:
COSMOS/Works = 0.01154 in
Theoretical (Singularity) = 0.01083 in
Percent Different = 6.60% (smaller than FEA)

\[
K_{arm\_rocker\_t} = \frac{100}{0.01154} = 8,665.51 \frac{lb}{in}
\]
In the case of axial loading of a constant cross-section part, the displacement is simply:

\[ y = \frac{FL}{AE} \]

- \( y \) = deflection
- \( F \) = Force applied axially
- \( L \) = Length of the part = 11.75 in
- \( A \) = Cross-section area = \( \pi \left( 0.1875^2 - 0.125^2 \right) = 6.136 \times 10^{-2} \)
- \( E \) = Modulus of Elasticity

When \( F = 100 \)

\[ y = \frac{100 \times (11.75)}{6.136 \times 10^{-2} \times (10,007,603)} = 1.914 \times 10^{-3} \]

Deflection at the end of the tube with a 100 lb force applied is \( 1.914 \times 10^{-3} \text{ in} \)

COSMOSWorks calculated the deflection at the end of the Tube to be 0.001906. The difference is

\[ \frac{0.001906 - 0.001914}{0.001914} \times 100 = -0.42\% . \]

In conclusion, the results are as follows:
- COSMOS/Works = 0.001906 in
- Theoretical (Singularity) = 0.001914 in
- Percent Different = 0.42\% (larger than FEA)

\[ K_{\text{tube}} = \frac{100}{1.914 \times 10^{-3}} = 52,261 \frac{\text{lb}}{\text{in}} \]
Rod End
By simplifying the rod end into multiple sections and perform analysis, it would be possible to combine the stiffness constant of each of these segments into a lumped stiffness constant. The rod end was divided into three segments, as shown below.

The constraint for Rod End 1 was immovable on the cylindrical surface and 100 lb force applied at the location where the segment connects to Rod End 3. The same constraint was applied to Rod End 2 with the exception that the larger cylindrical surface was left unconstrained because the head of the shoulder screw does not touch the surface. Force of 100 lb was once again applied at the cross-section where the part connected to Rod End 3. Rod End 3 has 2 constraints where it comes into contact with Rod End 1 & 2. Those two surfaces were fixed and a force of 100 lb was applied along the axis of the cylindrical surface.

Their maximum deflections in the y-direction and their respective stiffness constants were:

- Rod End 1 = 2.156e-5 in \( K_1 = 4,638,219 \text{ lb} \)
- Rod End 2 = 6.395e-5 in \( K_2 = 1,563,721 \text{ lb} \)
- Rod End 3 = 3.368e-5 in \( K_3 = 2,969,121 \text{ lb} \)

Sections 1 and 2 have the same deflections therefore, they are spring in parallel and their combined stiffness is \( K_{12} = 4,638,219 + 1,563,721 = 6,201,940 \). While section 1 and 2 are connected to section 3, the force passing through them are the same and therefore they are springs in series and the effective spring in series equation is:

\[
K_{\text{combine}} = \frac{K_{12}K_3}{K_{12} + K_3} = \frac{6,201,940 \times 2,969,121}{6,201,940 + 2,969,121} = 2,007,871
\]
The deflection of the entire rod end with the same constraints and force applied resulted in a total deflection of:

\[
\text{Rod End} = 6.169\times10^{-5} \text{ in} \quad K_{\text{rodend}} = 1,621,008 \text{ lb/in}
\]

The net difference between the divided rod end and the non-divided rod end is

\[
\frac{1,621,008 - 2,007,871}{2,007,871} \times 100 = 19.27\% .
\]

This means that the divided rod end probably does not have the appropriate boundary conditions to replicate that seen in the non-divided version. Therefore, it would be ideal to use the smaller stiffness constant to prevent an overestimation of the stiffness constant.
In conclusion, the results are as follows:
COSMOS/Works Divided = 2,007,871 in
COSMOS/Works Non-divided = 1,621,008 in
Percent Different = 19.27% (Smaller than Non-divided)

\[ K_{rod\_end} = 1,621,008 \frac{lb}{in} \]
Rod End Small

The same boundary conditions were applied to the Rod End Small and finite element analysis was run and the result was:

\[ K_{rod\_end\_small} = 2,436,647 \frac{lb}{in} \]
Hard-Stop

The stiffness of the hard-stop was obtained experimentally. A known force of 150 lb was applied on the force transducer as seen in the Figure on the left while a dial indicator was placed under the hard-stop. The deflection obtained was:

Hard-Stop = 0.0025 in

Stiffness constant of the hard-stop

\[ k_{hs} = \frac{152}{0.0025} \Rightarrow k_{hs} = 60,800 \frac{lb}{in} \]
Over-Travel Mechanism Assembly

There are two “stiffnesses” for Bottom rod end sub-assembly. The rising motion of the cam results in a force from the bottom rod end being exerted on the enclosure sleeve. During the fall motion of the cam, the washer is exerting force on the spring which in turn exerts the same force on the enclosure sleeve to pull the THK cart down.

In order to determine each of the stiffnesses necessary to insert into Simulink, it is important to obtain the lumped stiffness of the pushing and pulling motion. The pushing motion requires the stiffnesses of screw attaching the bottom rod end to the left side of the arm rocker, bottom rod end, enclosure sleeve, adaptor, top rod end and the screw attaching the rod end to the impact block.

Finite element analyses were performed on these parts and are shown in the order given above.

Pushing Bottom Screw:

The bottom screw is fasten to the arm rocker and the constraint was applied at the cylindrical surface at which it is fasten. A bearing load of 100 lb was applied over the
area where the bottom rod end exerts. The deflection needed to determine the stiffness of the bottom screw should be obtained at the center of the applied force. The range observed is 3.990e-5 to 3.692e-4 and the average between the max and min is 2.0455e-4 in. This value will be used to determine the stiffness of the bottom screw.

\[ K_{\text{bottom screw}} = \frac{488,878}{\text{lb/in}} \]

**Bottom Rod End:**

The bottom rod end was constrained at the location where the bottom rod end meets the enclosure sleeve. A bearing force of 100 lb was applied at the cylindrical surface that mates with the bottom screw. The maximum deflection is 3.811e-5 in. The stiffness constant of the bottom rod end is:

\[ K_{\text{bottom rod end}} = \frac{2,623,983}{\text{lb/in}} \]

The bottom rod end was constrained at the location where the bottom rod end meets the enclosure sleeve. A bearing force of 100 lb was applied at the cylindrical surface that mates with the bottom screw. The maximum deflection is 3.811e-5 in. The stiffness constant of the bottom rod end is:

\[ K_{\text{bottom rod end}} = \frac{2,623,983}{\text{lb/in}} \]
Enclosure sleeve:

The enclosure sleeve was constrained to be similar to it fastened to the adaptor and a force of 100 lb was applied over the area that the rod end would be exerting on it. The maximum deflection was determined to be 2.394e-4 in. The stiffness constant of the enclosure sleeve is:

\[ K_{\text{enclosure sleeve}} = 417,711 \frac{lb}{in} \]
The surface that the adaptor touches the top rod end was fixed and a force of 100 lb was applied over the area that the adaptor would come into contact with the enclosure sleeve. The maximum displacement was found to be 1.859e-5 in and the stiffness constant is:

\[ K_{\text{adaptor}} = 5,379,236 \frac{lb}{in} \]

Top rod end:
The cylindrical surface of the top rod end was constrained and a force of 100 lb was applied at the location where the adaptor contacts the rod end. The maximum displacement was 3.371e-5 in. The stiffness constant of the top rod end is:

$$K_{\text{top\_rod\_end}} = \frac{2,966,479 \text{ lb}}{\text{in}}$$

The combined stiffness for the pushing motion would be stiffnesses in series of bottom screw + bottom rod end + enclosure sleeve + adaptor + top rod end + top screw. The equation of the lumped stiffness for the pushing motion would be:

$$\frac{1}{K_{\text{lump}}} = \frac{1}{K_{\text{bs}}} + \frac{1}{K_{\text{bre}}} + \frac{1}{K_{\text{es}}} + \frac{1}{K_{\text{adaptor}}} + \frac{1}{K_{\text{tre}}} + \frac{1}{K_{\text{ts}}}$$

$$K_{\text{bs}} = 488,878, K_{\text{bre}} = 2,623,983, K_{\text{es}} = 417,711, K_{\text{adaptor}} = 5,379,236, K_{\text{tre}} = 2,966,479$$

$$\frac{1}{K_{\text{lump}}} = \frac{1}{488,878} + \frac{1}{2,623,983} + \frac{1}{417,711} + \frac{1}{5,379,236} + \frac{1}{2,966,479} + \frac{1}{488,878}$$

$$\frac{1}{K_{\text{lump}}} = 2.045e - 6 + 3.811e - 7 + 2.394e - 6 + 1.859e - 7 + 3.371e - 7 + 2.045e - 6$$

$$\frac{1}{K_{\text{lump}}} = 7.388e - 6 \Rightarrow K_{\text{lump\_push}} = 135,353 \frac{\text{lb}}{\text{in}}$$
Pulling
The pulling motion consists of the deflection of the bottom screw, bottom rod end, shoulder screw and washer, spring, enclosure sleeve, adaptor, top rod end, and the top screw.

Bottom & Top screw:
\( K_{bs} \) & \( K_{ts} = 488,878 \)

Bottom rod end:
Displacement = 5.769e-5 in
\( K_{bre} = 1,733,403 \)

Shoulder screw & Washer:
Displacement = 9.710e-5
\( K_{ssw} = 1,029,866 \)

Spring: \( K_{spring} = 225 \)

Enclosure sleeve:
Displacement = 1.585e-4
\( K_{es} = 630,915 \)

Adaptor:
Displacement = 3.411e-5
\( K_{adaptor} = 2,931,692 \)

Top rod end:
Displacement = 2.786e-5 in
\( K_{bre} = 3,589,376 \)

\[
\frac{1}{K_{lump\_pull}} = \frac{1}{K_{bs}} + \frac{1}{K_{bre}} + \frac{1}{K_{ssw}} + \frac{1}{K_{spring}} + \frac{1}{K_{es}} + \frac{1}{K_{adaptor}} + \frac{1}{K_{tse}}
\]

\[
\frac{1}{K_{lump\_pull}} = \frac{1}{488,878} + \frac{1}{1,733,403} + \frac{1}{1,029,866} + \frac{1}{225} + \frac{1}{630,915}
\]

\[
+ \frac{1}{2,931,692} + \frac{1}{3,589,376} + \frac{1}{488,878}
\]

\[
\frac{1}{K_{lump\_pull}} = 2.04e - 6 + 5.77e - 7 + 9.71e - 7 + 4.44e - 3 + 1.59e - 6 + 3.41e - 7
\]

\[
+ 2.79e - 7 + 2.04e - 6
\]

\[
\frac{1}{K_{lump\_pull}} = 4.45e - 3 \quad \Rightarrow \quad K_{lump\_pull} = 224.8
\]
Appendix F: Lumped Stiffness Constants Calculation

\[ k_{la} = 12,150 \text{ lb/in} \]

\[ k_{cr} = \frac{1}{k_{re} + \frac{1}{k_{tube}} + \frac{1}{k_{res}}} = \frac{1}{1,621,008 + \frac{1}{52,261} + \frac{1}{2,436,647}} = 49,598 \text{ lb/in} \]

\[ k_{arr} = 850 \text{ lb/in} \]

\[ k_{arl} = 8,665 \text{ lb/in} \]

\[ k_{bre} = 135,353 \text{ lb/in (pushing)} & 224.8 \text{ lb/in (pulling)} \]

Combined stiffness constants:

\[ k_i = \frac{k_{la} \times k_{cr}}{k_{la} + k_{cr}} = \frac{12,150 \times 49,598}{12,150 + 49,598} = 9,755 \frac{\text{lb}}{\text{in}} \]
\[ k_2 = k_{arr} = 850 \frac{lb}{in} \]

\[ k_{3\text{ pushing}} = \frac{k_{arl} \times k_{bre}}{k_{arl} + k_{bre}} = \frac{8,665 \times 135,353}{8,665 + 135,353} = 8,144 \frac{lb}{in} \]

\[ k_{3\text{ pulling}} = \frac{k_{arl} \times k_{bre}}{k_{arl} + k_{bre}} = \frac{8,665 \times 224.8}{8,665 + 224.8} = 219 \frac{lb}{in} \]

Transferring \( k_2 \) over to the end effector side and combining it with \( k_3 \):

\[
K_2 = \left( \frac{a}{b} \right)^2 k_2 = \left( \frac{7.75}{2.375} \right)^2 850 = 9,051 \frac{lb}{in}
\]

Transferring \( k_1 \) over to the end effector side:

\[
K_1 = \left( \frac{a}{b} \right)^2 k_1 = \left( \frac{7.75}{2.375} \right)^2 9,755 = 103,873 \frac{lb}{in}
\]

Masses:

\[
M_1 = 16.351 \text{ lb} \quad 7.417 \text{ kg}
M_2 = 1.6388 \text{ lb} \quad 0.743 \text{ kg}
M_3 = 0.3854 \text{ lb} \quad 0.175 \text{ kg}
\]

Stiffnesses:

\[
K_1 = 103,873 \text{ lb/in} \quad 18,190,949 \text{ N/m}
K_2 = 9,051 \text{ lb/in} \quad 1,585,073 \text{ N/m}
K_3 = 8,144 \text{ lb/in} \quad 1,426,233 \text{ N/m} \quad \text{(Pushing)}
\]

or

\[
219 \text{ lb/in} \quad 38,353 \text{ N/m} \quad \text{(Pulling)}
\]
UNLESS OTHERWISE SPECIFIED: TOLERANCES:

3x Ø2.5 THRU M3 x 0.5mm-6G THRU

SCALE: 1:1

Ø 5/16 1.8125
3/8-16 1.50

*Note: Dimensions are in inches unless noted otherwise.

**Stanchion RIB**
**Note:** Dimensions are in inches unless noted otherwise.

<table>
<thead>
<tr>
<th>UNLESS OTHERWISE SPECIFIED</th>
<th>NAME</th>
<th>DATE</th>
</tr>
</thead>
<tbody>
<tr>
<td>DIMENSIONS ARE IN INCHES</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TOLERANCES:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>FRACTIONAL:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ANGULAR: 3 DEGREES</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BEND: 12 DEGREES</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TWO PLACE DECIMAL:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>THREE PLACE DECIMAL:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>INTERPRET TOLERANCES</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TOLERANCING TEST:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MATERIAL: ALUMINUM</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NOT ASSY</td>
<td></td>
<td></td>
</tr>
<tr>
<td>USED ON</td>
<td></td>
<td></td>
</tr>
<tr>
<td>FINISH</td>
<td></td>
<td></td>
</tr>
<tr>
<td>APPLICATION</td>
<td></td>
<td></td>
</tr>
<tr>
<td>DO NOT SCALE DRAWING</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**TITLE:** Impact Block

**SIZE** | **DWG. NO.** | **REV.** |
----------|--------------|---------|
A         |              | 2       |

**SCALE:** 1:1  **WEIGHT:**  **SHEET:** 1 OF 1
Adapter

5/8"-18

5/16" 7/16" 3/4" Hex

1"

#10-32