PARTIAL COHERENCE AND OPTICAL VORICES

by

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A Dissertation
Submitted to the Faculty
of the
WORCESTER POLYTECHNIC INSTITUTE
in partial fulfillment of the requirements
for the Degree
of Doctor of Philosophy
in
Physics
by

June 9th, 2004

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ABSTRACT

Optical vortices are singularities in phase fronts of optical beams. They are characterized by a dark core in the center and by a helical wave front. Owing to azimuthal components of wave vectors, an optical vortex carries orbital angular momentum. Previously, optical vortices were studied only in coherent beams with a well-defined phase. The object of this dissertation is to explore vortices in partially coherent systems where statistics are required to quantify the phase. We consider parametric scattering of a vortex beam and a vortex placed on partially coherent beam. Optical coherence theory provides the mathematical apparatus in the form of the mutual coherence function describing the correlation properties of two points in a beam. Experimentally, the wave-front folding interferometer allows analysis of the cross-correlation function, which may be used to study partial coherence effects even when traditional interferometric techniques fail. We developed the theory of composite optical vortices, which can occur when two coherent beams are superimposed. We then reported the first experimental observation of vortices in a cross-correlation function (which we call spatial correlation vortices). We found numerically and experimentally how the varying transverse coherence length and position of a vortex in a beam may affect the position and existence of spatial correlation vortices. The results presented in this thesis offer a better understanding of the concept of phase in partially coherent light. The spatial correlation vortex presents a new tool to manipulate coherence properties of an optical beam.
ACKNOWLEDGMENTS

This PhD would not be possible without several individuals, few groups and a bit of luck. The foundation of what I know about physics and math come from my professors at the Moscow Institute of Physics and Technology. That basis was fortified and developed by my mentors at the Worcester Polytechnic Institute and at the Optical Sciences Center, University of Arizona.

The theory is nothing without experiment. And so I’d like to thank my advisor, Dr. Grover A. Swartzlander, Jr., for guiding me through all those years. His thorough approach to physics is undoubtedly cast on me, from conducting experiments to presenting results, albeit I’ll probably never reach his “degree of precision”. I would also like to thank him for the patience and last but not least, financial support.

I’m also grateful to the members of my Graduate Committee, Dr. Padmanabhan K. Aravindand and Germano S. Iannacchione for reviewing this work. My grateful acknowledgments go to WPI Physics department and its Head, Dr. Thomas H. Keil. They always provided me with financial and logistical support. However the best thing about WPI Physics, in my opinion, is the size of the department. I always felt a part of the community where people actually know and care about what you are doing. My gratitude also goes to the Optical Sciences Center, where I spent last 2.5 years, and its Director, Dr. James C. Wyant, who has always been extremely kind to me. Among the number of
people at OSC I’d like to especially acknowledge the help of Dr. Arvind Marathay, who
taught me coherence theory and whose advises were truly invaluable on so many
occasions.

Over these years I’ve always been very fortunate to have fellow graduate students
who I talked to on an everyday basis, exchanging and discussing ideas, brainstorming and
gaining from their experience. I’d like to especially name here David Rozas, Anton
Deykoon and David Palacios. Their contribution to this work is definitely not less than
my own.

Finally, I can say that this PhD took a lot of time. Lots of things changed over
these years including the focus of the work. And so I can now acknowledge that it would
never be finished without my wife, Inga. Thank you.
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### SYMBOLS AND ABBREVIATIONS

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<tr>
<td>OV</td>
<td>Optical Vortex</td>
</tr>
<tr>
<td>SCV</td>
<td>Spatial Correlation Vortex</td>
</tr>
<tr>
<td>Γ</td>
<td>Mutual Coherence Function (MCF)</td>
</tr>
<tr>
<td>( \vec{K} )</td>
<td>wavevector</td>
</tr>
<tr>
<td>( \vec{k} )</td>
<td>transverse wavevector</td>
</tr>
<tr>
<td>( \lambda )</td>
<td>wavelength</td>
</tr>
<tr>
<td>( r, \phi )</td>
<td>polar coordinates in ( x )-space</td>
</tr>
<tr>
<td>( x, y )</td>
<td>Cartesian coordinates in ( x )-space</td>
</tr>
<tr>
<td>( k_{\perp} )</td>
<td>transverse polar coordinate in ( k )-space</td>
</tr>
<tr>
<td>( k_x, k_y )</td>
<td>Cartesian coordinates in ( k )-space</td>
</tr>
<tr>
<td>( \pi )</td>
<td>3.141592654</td>
</tr>
<tr>
<td>( E )</td>
<td>electric field</td>
</tr>
<tr>
<td>( E_0 )</td>
<td>electric field amplitude</td>
</tr>
<tr>
<td>( I )</td>
<td>intensity</td>
</tr>
<tr>
<td>( t )</td>
<td>time</td>
</tr>
<tr>
<td>( m )</td>
<td>topological charge of an optical vortex</td>
</tr>
<tr>
<td>( w )</td>
<td>size of a generic beam</td>
</tr>
<tr>
<td>( w_0 )</td>
<td>size of a Gaussian beam</td>
</tr>
<tr>
<td>( i )</td>
<td>imaginary unit, ( i = \sqrt{-1} )</td>
</tr>
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1. INTRODUCTION

Vortices are fascinating topological features which are ubiquitous throughout the Universe. They may be found in many branches of physics ranging from superfluidity\(^1\) to aero- and hydro-dynamics\(^2\) and global weather patterns\(^3\). Over the last two decades, potential applications raised interest in vortices within optical fields\(^4\). Optical vortices (OVs) were found to occur in coherent radiation such as Laguerre-Gaussian laser beams\(^5\), light scattered from rough surfaces\(^6, 7\), optical caustics\(^8, 9\) and OV solitons\(^10, 11\). Possible applications of optical vortices include optical switching\(^12\), optical trapping\(^13, 14\) and coherence filtering\(^15, 16\).

OVs are phase objects and require a degree of correlation between different points in the beam. Coherent OVs are also known to carry an associated orbital angular momentum. This thesis analyzes OVs in the partially coherent light, where the statistical properties of light should be taken into account to quantify global phase properties\(^17, 18\). A brief introduction to major definitions and a historical account concerning optical vortices and partial coherence is given in Chapter 2.

The partial coherence effects may be investigated in two ways. The main approach, which is the focus of this thesis, starts with the partially coherent light source. The OVs are then created by passing a partially coherent beam through a phase mask, and the resulting partially coherent vortex beam is investigated. The problem of detection of optical vortices in partially coherent light, where the traditional interferometric techniques may fail, is addressed in Chapter 3. We use a vortex beam, truncated at the
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waist⁴ and experimentally analyze the far-field distribution, where an asymmetrical intensity distribution is expected due to the helicity of a vortex wavefront.

The presence of OVs may be expected to change the coherence properties of a source. The main contribution of this author and his collaborators is the investigation of optical singularities, which may form in the cross-correlation function of the resulting beam. For the first time we detect experimentally, and investigate theoretically, this new type of optical vortices, named spatial correlation vortices. Our ability to selectively change the spatial coherence properties of a light beam may potentially help to develop a new branch of applications for optical vortices. Examples include optical coherence tomography, where control of the spatial coherence can suppress the coherent scattering from the undesirable areas, and coherence filtering. The groundwork is presented in Chapter 4, where we discuss composite optical vortices that may be created in the coherent regime when two collinear optical vortex beams are superimposed¹⁹. Mathematical modeling and numerical calculation results for the partial coherence case are presented in Chapter 5. Experimental investigation of spatial correlation vortices is presented and discussed in Chapter 6.

The second approach to the study of partial coherence involves passing a coherent beam of interest (e.g. optical vortex) through an optical system which affects coherence properties of a beam, for example a diffuser²⁰. Optical down-conversion, also involving the non-linear transformation of a pump photon into a signal and an idler may serve as an example. The statistics of the signal and the idler beams, produced by down-conversion of a vortex pump beam, are presented in Chapter 7.
Parts of the thesis have been published in or submitted to the following journals:


and were presented at the following international conferences:


2. HISTORY

2.1 Optical vortex

Optical vortices (OV’s) occur as particular solutions to the wave equation in cylindrical coordinates. The essential characteristic of a field vortex is the helical phase front. For a single vortex (see Fig. 2.1), centered on the optical axis, the transverse phase profile is given by $\Phi(r,\phi) = m\phi$, where $(r,\phi)$ are the transverse polar coordinates, and $m$ is a signed integer called the “topological charge”. In the following description we assume the non-trivial case $m \neq 0$. The center of a vortex for a coherent beam is characterized by a dark core [for $m=1$ see Fig. 2.1(a)], within which the intensity vanishes at a points. As one travels a full circle around the core, the phase of the electric field changes by an integer factor of $2\pi$ (measured in radians), as shown in Fig. 2.1(b). At the center of the core the phase is undefined; this is physically recognizable since the intensity vanishes at this point. For this reason, a vortex is sometimes called a phase singularity. Assuming the path of the vortex beam lies on the optical axis, $z$, and adding the initial phase $\beta$ of a field, which carries a vortex, and a phase factor, $-kz$, to describe the light propagating paraxially along the $z$-axis, using cylindrical coordinates $(r,\phi,z)$, we obtain the phase $\Phi(r,\phi,z) = m\phi + \beta - kz$, where $k$ is the wavenumber. In three dimensions, the surface of constant phase form helical spirals with a pitch of $m\lambda$, where $\lambda$ is the vacuum wavelength. Helicity of the phase front means that the wavevectors have azimuthal
components that circulate around the core\textsuperscript{22}. Owing to this circulation, the optical wave carries orbital angular momentum\textsuperscript{23}.

The pioneering work on helical-wavefront waves was reported by Bryngdahl\textsuperscript{24, 25}, while the actual term “optical vortex” was offered by Coullet et al.\textsuperscript{26}. Screw and edge dislocation in optical wavefronts were analyzed in detail by Nye and Berry\textsuperscript{6, 8, 27, 28} and Wright\textsuperscript{29}. Screw phase dislocations were first observed in laser output beams\textsuperscript{30} and one of the most familiar examples of OV’s is the TEM*$_{01}$ “doughnut” mode of a laser cavity\textsuperscript{26, 31-34} (shown on Fig. 2.1). Multiple vortex modes in various resonators were also investigated; propagation dynamics, stable spatial structure and creation and annihilation of vortex modes have all been extensively explored\textsuperscript{35-45}. Field vortices were also found in laser speckle patterns, produced by surface scattering of optical waves\textsuperscript{7, 46}. Beams with spiral wavefronts were considered theoretically by Kruglov and Vlasov\textsuperscript{47} and by Abramochkin and Volostnikov\textsuperscript{48}.

Experimental means to obtain optical vortices include active laser systems\textsuperscript{30, 38, 49, 50}, low-mode optical fibers\textsuperscript{51}, alteration of the beam wavefront with a phase mask\textsuperscript{10, 52}, computer-generated holography\textsuperscript{53-57}, and a mode-converter\textsuperscript{58-60}. Optical vortices were also found to occur in non-linear optics\textsuperscript{9, 61-65}, where their most attractive feature is the possibility of creation of optical vortex solitons\textsuperscript{10, 11, 66-69}. Propagation of OV’s in self-defocusing media\textsuperscript{70, 71}, vortex creation via nonlinear instability\textsuperscript{66, 72, 73}, quantum nonlinear optical solitons\textsuperscript{74, 75}, polarization effects\textsuperscript{67}, propagation dynamics of vortices\textsuperscript{76-83}, and vortex arrays\textsuperscript{84, 85}, as well as other effect\textsuperscript{86}, were investigated. More recently, an interest in optical vortices was sustained by their similarities with quantised vortices in superfluids and Bose-Einstein condensates\textsuperscript{87-90}. 
2. HISTORY

A single vortex placed within a coherent scalar beam in a paraxial approximation may be expressed by the complex electric field:

\[
E(r,\phi,z,t) = A(r',z)g(r,z)\exp[i(m\phi' + \beta + 2\pi \epsilon / \lambda - \omega t)]
\]  

(2.1)

where coordinates \((r, \phi)\) and primed coordinates \((r', \phi')\) are measured with respect to the center of the beam and the vortex core, respectively, \(\omega\) is the angular frequency, \(t\) is the time, \(\lambda\) is the wavelength, \(g(r,z)\) and \(A(r',z)\) are respectively the beam envelope and the vortex core profile in the transverse \(xy\) plane at \(z\). The field phase \(\beta\) is an arbitrary constant. We will often consider the beam in the initial plane at \(z = 0\), where \(g(r)\) and \(A(r')\) will denote beam envelope and vortex core profile in that plane. Perhaps the most typical and mathematically simple description of a beam distribution are a Gaussian envelope: \(g(r) = E_0 \exp(-r^2/w_0^2)\), and a plane wave: \(g(r) = E_0\). Here, \(E_0\) and \(w_0\) are a measure of the field amplitude (assumed to be real) and characteristic beam size, respectively. The more common vortex core profiles belong to “large-core”52, 91-96, \(A(r') = |r' / w_0|^{m}\), and “small-core” (“point”) vortices57, \(A(r') = \tanh(|r' / w_s|).\) Here \(m\) again is the topological charge and \(w_s\) is the size parameter, \(w_s \ll w_0\). When the centers of the beam and the vortex core coincide, than we call the vortex an “on-axis” vortex, otherwise the vortex is said to be an “off-axis” vortex.

2.2 Detection of Optical Vortices

Owing to total destructive interference, a characteristic of an optical vortex is a point of zero intensity in the dark vortex core. However, to identify an optical vortex it is not sufficient to simply identify the points of zero intensity. Instead, one must obtain
information about the phase to verify that the phase is harmonic (when satisfied, this condition also satisfies the requirement that the intensity vanishes at the core).

One of the most common methods of detection of optical singularities in the coherent light\(^97\) is the interference with a reference wave. In a collinear wave interference, a spiral fringe pattern occurs \(^{53, 61, 91}\), or, in a case of equal wavefront curvatures, radial fringes\(^{24, 98}\). When interfering beams are non-collinear, the location and topological charge of a vortex may be identified by a characteristic forking pattern in the interference fringes\(^46\), shown on Fig. 2.1(c). For a vortex field 

\[
E(r, \phi) = E_0 A(r) \exp(i m \phi + i \beta)
\]

and a planar reference wave given by 

\[
E' = E_0' \exp(i k_z x)
\]

where \(k_z\) is the transverse wavenumber of the tilted plane wave, the interferogram has a profile given by 

\[
|E + E'|^2 = E_0^2 A^2(r) + E_0'^2 + 2E_0 E_0' A(r) \cos(m \phi + \beta + k_z x).
\]

At the center of the core (the origin) the intensity of the interferogram has a value of \(E_0'^2\). Above \((x \approx 0, y > 0)\) and below \((x \approx 0, y < 0)\) the phase singularity, the value of the factor \(\cos(m \phi + \beta + k_z x)\) switches sign, thereby creating a forking pattern. If we encircle the vortex core, then the number of fringes entering the circled area differs from the number of exiting fringes by \(m\). If we choose the circle around a positively charged vortex so that the number of entering fringes is 1, then inside the area that fringe splits into \(m+1\) fringes. Conversely, if vortex has a negative charge, then inside the circle \(m+1\) entering fringes will combine into a single fringe. Therefore analysis of interference fringes identifies location and topological charge of a vortex. A related split beam technique may be used when a coherent reference beam is not available\(^{31, 99}\). Other detection methods include the mode converter technique\(^{58, 59}\) and holography\(^{93, 100}\).
The above description of optical vortices assumes a well defined phase and a strict relation between phases at different points in the beam, i.e. different points in the beam are correlated or coherent. In the real world, however, both intensity and phase of any point of a beam may fluctuate randomly. In partially coherent light, where statistics are necessary to describe the phase, the description of vortices is more complicated. As a model, one may assume that the phase $\beta$ is varying randomly with time and location, and employ a correlator to describe the mutual coherence (correlation) of pairs of points in the beam.

2.3. Partial coherence

The concept of coherence may be traced back to the pioneering experiments by Young in 1807\cite{Young1807} and Fresnel in 1814-1816\cite{Fresnel1814,Fresnel1816,Fresnel1816-2,Fresnel1816-3} on the interference of light from two pinholes. The fundamental result of the experiment was the understanding that in order to observe the interference of light, the two pinholes must be illuminated by a single source. If each pinhole is illuminated by an independent source than the phase difference between two sources is generally a random value. The changes in phase difference happen so fast that the typical integration measurement tool (eye, film, or camera sensor) registers a certain average of the instantaneous interference patterns. This may happen when light comes from two different points of the same source or for light coming from a single point but at different instants of time. If no interference is observed then such non-interfering beams are called \textit{incoherent}. In Young’s original experiment, the conditions of small path difference and smallness of the source were both taken into account to create two beams capable of interfering. The term “coherent” was used to describe interfering beams.
To quantify the coherence properties the concept of coherence volume may be used. Here we only give a brief account of the necessary definition and refer the reader to the appropriate textbooks. \(^{17,18}\) Interference fringes in Young’s experiment (see Fig. 2.1) were found to form only if \(\Delta \theta \Delta s \leq \overline{\lambda} \), where \(\Delta \theta\) is the angle that the distance between the pinholes subtends at the source, \(\Delta s\) is the source size, and \(\overline{\lambda}\) is the mean wavelength.

If \(R\) is the distance between the plane containing an incoherent (e.g. thermal) source and the plane containing the pinholes (see Fig. 2.1), than in order to observe the interference fringes, the pinholes must be located within a region around the axial point \(Q\) in the pinhole plane. This region is called the coherence area of light in the pinhole plane and its area is on the order of \(a_c \approx (R \Delta \theta)^2\). The square root of the coherence area is \(l_c\), sometimes called the transverse coherence length. On the other hand, interference fringes are not observed when the time delay between two interfering beams coming from the same source is greater than the coherence time, \(\tau_c\), given by the order-of-magnitude relation \(\tau_c \approx 1/\Delta \nu\) with the \(\Delta \nu\) being the spectral width. The corresponding longitudinal coherence length is defined by \(L_c = c \tau_c \approx c / \Delta \nu = \overline{\lambda}^2 / \Delta \lambda\). The volume of a cylinder with the bottom area equal to the coherence area, and the height equal to the \(L_c\), defines the coherence volume.

The understanding of partial coherence came into optics as a result of the work of Van Cittert, who calculated the partition functions, showing the correlation between amplitudes in different points,\(^ {105,106}\) and Zernike, who actually introduced the notion of degree of coherence and mutual intensity.\(^ {107}\) Several authors made major contributions to the development of the concept of coherence in the 1950’s, including Hopkins,\(^ {108,109}\)
Parrent\textsuperscript{110}, and Wolf\textsuperscript{111, 112}. The concepts of both the second-order coherence (correlation between points in the field)\textsuperscript{113, 114} and the forth-order correlation between photons (correlation of intensities)\textsuperscript{115-117} emerged. The development of the coherence theory rapidly continued with the invention of the coherent source (laser) and led to the creation of the quantum theory\textsuperscript{118-120} for optics, which has now developed into the field of quantum optics\textsuperscript{121}. Probably one of the first summaries of coherence properties of optical fields is the review by L. Mandel and E. Wolf \textsuperscript{122}. The coherence theory has developed further with numerous papers published and several textbooks available\textsuperscript{17, 18, 123, 124}. In this thesis own main interest is the application of coherence theory to optical vortices. The methods of quantitative description of the coherence, the characteristics of the partially coherent light sources, and the propagation equations for the partially coherent beams, will be central in this description.

2.4. Mutual coherence function

We are normally concerned with stationary fields. Moreover as a rule the fields are also assumed ergodic, i.e. time and ensemble averages are equivalent. The mathematical description of the second-order coherence theory is based on the mutual coherence function (MCF)\textsuperscript{18, 112}, which may be defined as:

\[ \Gamma(\vec{r}_1, \vec{r}_2, \tau) = \langle E_1(\vec{r}_1, t + \tau)E_2^*(\vec{r}_2, t) \rangle \]  

(2.2)

where \( \langle \rangle \) denotes the averaging over time \( t \), and \( \tau \) is the time delay between fields \( E_1 \) and \( E_2 \) at transverse plane points \( \vec{r}_1 \) and \( \vec{r}_2 \) respectively. The subscript \( j=1, 2 \) on the electric field \( E_j \) emphasizes that the field is a function of \( \vec{r}_j \). The MCF is a
multidimensional function and may often by characterized by two derivative functions: 
the [average] intensity:

\[ I(\vec{r}) = \Gamma(\vec{r}, \vec{r}, 0) \]  

(2.3)

and the [average] cross-correlation:

\[ X(\vec{r}) = \Gamma(\vec{r}, \vec{r}, 0) \]  

(2.4)

From a physical standpoint, it is convenient to express MCF as a product of the field
\( E_1(\vec{r}_1) \), its complex conjugate field \( E_2^*(\vec{r}_2) \), and a correlator \( C(\vec{r}_1, \vec{r}_2) \): 

\[ \Gamma(\vec{r}_1, \vec{r}_2) = E_1(\vec{r}_1) E_2^*(\vec{r}_2) C(\vec{r}_1, \vec{r}_2) \]  

(2.5)

Alternatively, MCF may be expressed in the form:

\[ \Gamma(\vec{r}_1, \vec{r}_2, \tau) = I(\vec{r}_1)^{1/2} I(\vec{r}_2)^{1/2} \gamma(\vec{r}_1, \vec{r}_2, \tau) \]  

(2.6)

where \( \gamma(\vec{r}_1, \vec{r}_2, \tau) = \Gamma(\vec{r}_1, \vec{r}_2, \tau)/[I(\vec{r}_1)I(\vec{r}_2)]^{1/2} \) is the normalized MCF, also known as the 
complex degree of coherence\textsuperscript{18}.

The description of spectral effects of partial coherence\textsuperscript{125,126} may be based on the
cross-spectral density \( W(\vec{r}_1, \vec{r}_2, \omega) \) (CSD), defined by the equation\textsuperscript{18}:

\[ < \tilde{E}_1(\vec{r}_1, \omega) \tilde{E}_2^*(\vec{r}_2, \omega') > = W(\vec{r}_1, \vec{r}_2, \omega) \delta(\omega - \omega') \]  

(2.7)

where \( \tilde{E}_1(\vec{r}_1, \omega) \) and \( \tilde{E}_2(\vec{r}_2, \omega') \) are electric fields having frequencies \( \omega \) and \( \omega' \) in the
angular frequency domain (the wave sign on top of a field specifies the frequency
domain, transformation between time and frequency domains is described by a Fourier
integral\textsuperscript{18}), \( \delta(\omega - \omega') \) is the delta-function.
2.5. Partially coherent sources

One of the cornerstones of coherence theory, and optics in general, is the description of the source of light. Coherence properties of the source may be described with the spatial coherence function \( \Gamma(\vec{r}_1, \vec{r}_2) \), where we consider all points of the source at the same instant of time and drop \( \tau = 0 \) from notation. We usually restrict our attention to the 2-dimensional planar sources\(^{127-129} \), although 3-dimensional sources have also been investigated\(^{130,131} \).

In recent years, the so-called Schell-model sources\(^{132,133} \) have been playing an increasingly important role in optical coherence theory\(^{134-140} \). For a Shell-model source, the complex degree of coherence between two points depends on \( \vec{r}_1 \) and \( \vec{r}_2 \) only through the difference \( \vec{r}_1 - \vec{r}_2 \): \( \gamma(\vec{r}_1, \vec{r}_2) = \gamma(\vec{r}_1 - \vec{r}_2) \). For modeling purposes in this thesis, we routinely use sources with the Gaussian-Shell correlator\(^{141} \), where the Schell-model type correlator has a Gaussian distribution:

\[
C(\vec{r}_1 - \vec{r}_2) = \exp\left(-\left|\vec{r}_1 - \vec{r}_2\right|^2 / l_C^2 \right)
\]

and \( l_C \) is the transverse coherence length.

Further investigation has led to the development of the Carter-Wolf source\(^{128} \), where MCF is expressed in the form: \( \Gamma(\vec{r}_1, \vec{r}_2) = I(\vec{r}_1 + \vec{r}_2) \gamma(\vec{r}_1 - \vec{r}_2) \). Here the intensity is a function of the sum of coordinates \( \vec{r}_1 \) and \( \vec{r}_2 \), and \( \gamma \) is a function of \( \vec{r}_1 - \vec{r}_2 \). The separable intensity and \( \gamma \) of a Carter-Wolf source greatly simplify the propagation analysis of the MCF. For example, by making transition to the average and difference variables, \( \vec{r} = \vec{r}_1 + \vec{r}_2 \) and \( \vec{r}_{12} = \vec{r}_1 - \vec{r}_2 \), the spatial Fourier transform of the MCF becomes a product
of Fourier transforms of intensity and $\gamma$. The limitation of the Carter-Wolf source is the requirement for the intensity profile to be broad compared to that of spatial coherence $^{128}$.

2.6. MCF investigation, vortex

The possibility of obtaining non-diffracting sources $^{137, 142}$ sparked further interest in the partially coherent sources. Bessel Schell-model sources produced with circular apertures were analyzed using modal expansion $^{143}$, followed by more thorough study of partially coherent Bessel-Gauss beams with a Gauss-Schell complex degree of coherence $^{144}$. The next step was the application of modal expansion to partially coherent sources with helical modes $^{145}$ and investigation of their propagation $^{146}$. Partially coherent beams with separable phase, carefully constructed to carry an optical vortex modes, have been recently studied analytically $^{147}$ and experimentally $^{148}$.

Recently, the actual structure of the MCF and CSD has been investigated. For example, the phase singularities in the CSD were analyzed using partially coherent light emerging from two pinholes $^{149}$. Modal expansion was used to predict vortices in CSD $^{150}$. Spatial Correlation Vortices (SCV’s) in the cross-correlation function of a partially coherent beam with an off-axis vortex were analyzed numerically $^{151}$, analytically $^{152}$ and confirmed experimentally $^{153}$. In the case of a partially coherent beam with an on-axis vortex, the dislocation ring in the cross-correlation was observed even for low-coherent light $^{154}$. Thus, the cross-correlation proved to be a more robust description of coherence effects in singular optics than the intensity.
2.7. Propagation, beam spread

The coherence properties of a beam may change upon propagation\textsuperscript{107, 111, 112}. Most notably the light, propagated from an incoherent source, becomes partially coherent\textsuperscript{18}. That result is the essence of Van Cittert-Zernike theorem\textsuperscript{107}. A real life example of partial coherence is the image of a mesh fence, composed of thin wires and illuminated by a incoherent source of small, but non-zero angular size (sunlight or a thermal source, placed at a sufficiently large distance from the wires). The light at the fence is partially coherent. The reader may readily verify that although the contrast of an image on a screen decreases as we move the screen away from the fence due to diffraction\textsuperscript{155}, the wires become clearly visible when the screen is placed close to a fence at a distance much greater than the wire thickness. In comparison, wires are not visible when an incoherent light source (f. e. a flashlight covered with a diffuser) is placed directly behind the fence.

The effect of wire visibility is most easily understood by considering the angular beam spread\textsuperscript{156-159}, which describes how the major portion (fraction $\varepsilon=0.96$) of the beam power is spread as the distance from the source is increased\textsuperscript{17}. For a Gaussian Carter-Wolf source, characterized by $I(\vec{r}_1 + \vec{r}_2) = I_G \exp(-|\vec{r}_1 + \vec{r}_2|^2 / 2w_0^2)$ and $\gamma(\vec{r}_1 - \vec{r}_2) = \exp(-|\vec{r}_1 - \vec{r}_2|^2 / 2l_C^2)$, the radius of a circle that encloses $\sim96\%$ of the beam power is given by the equation

$$R_{Cw} = [2\pi w_0^2 (1 + z^2 / k^2 w_0^2 l_C^2)]^{1/2} \quad (2.9)$$
where $z$ is the distance from the source, $w_0$ is the Gaussian beam size at the source, and $k$ is the propagation constant (modulus of the wave-vector). The assumption $w_0 \gg l_c$ is used. In the limit $z \gg k w_0 l_c$ (far from the source), the ratio of $R_{Cw}$ to $z$ is inversely proportional to the coherence length:

$$R_{Cw} / z \approx (2\pi)^{1/2} / kl_c$$

(2.10)

On the other hand, next to the incoherent source the light goes in all directions and $R_{Cw} / z \approx (2\pi)^{1/2} w_0 / z$, i.e. for small value of $z$ the ratio of $R_{Cw}$ to $z$ may be infinitely large (incoherent beam).

The mathematical description of the propagation of partially coherent beams is largely based on the Rayleigh-Sommerfeld diffraction theory. The Rayleigh diffraction formula of the first kind, derived in 1897, provides the solution to the Dirichlet boundary problem:

$$E(\vec{r},z) = -1/2\pi \iint_{z=0} E(\vec{r}',0) \frac{\partial}{\partial z} \{\exp(ikR) / R\} d\vec{r}'$$

(2.11)

where $E(\vec{r},z)$ is the field at a point $\vec{r}$ in the transverse plane $z \geq 0$ and $E(\vec{r}',0)$ is the field at a point $\vec{r}'$ in the transverse plane $z=0$, $k$ is the wave number, and $R$ is the distance between points $(\vec{r}',0)$ and $(\vec{r},z)$ (see Fig 2.2). At the limit $R \gg \lambda$ ($1 / R \ll k$) in a paraxial approximation Eq. (2.11) turns into the diffraction integral:

$$E(\vec{r},z) = ik / 2\pi \iint_{z=0} E(\vec{r}',0) \{\exp(ikR) / R\} d\vec{r}'$$

(2.12)

Armed with that theory, Zernike derived a propagation equation for the mutual intensity in the paraxial approximation. More general derivation originating from Eq. (2.11) is due to Wolf and in the MCF notation may be expressed as (see e.g. Ch. 4.4):
\begin{equation}
\Gamma(\vec{r}_i, \vec{r}_2, z) = (k / 2\pi)^3 \int_{z=0}^{\infty} \Gamma(\vec{r}_i', \vec{r}_2', 0) \exp[ik(R_1 - R_2)]R_1^{-1}R_2^{-1} \cos \theta_1 \cos \theta_2 d\vec{r}_1 d\vec{r}_2
\end{equation}

where \( R_1 = |\vec{r}_1 - \vec{r}_i'|, \ R_2 = |\vec{r}_2 - \vec{r}_2'|, \) and \( \theta_1 \) and \( \theta_2 \) are the angles between positive z-direction and lines through pairs of points \((\vec{r}_1, \vec{r}_i')\) and \((\vec{r}_2, \vec{r}_2')\) respectively (see Fig. 2.2).

### 2.8. Spectral and temporal coherence effects

Spectral phenomena in vortex fields stem from the fact that the helicity of a vortex wavefront introduces a transverse \( k \)-vector component. If we place a vortex on a planar “seed” beam of wavelength \( \lambda_b \), the wavenumber \( K \) of a vortex beam differs from the wavenumber \( K_b \) of the seed beam. Hence, the wavelength of a vortex beam \( \lambda = 2\pi/K \) differs from \( \lambda_b \). Recently, color effects were studied theoretically by Berry\textsuperscript{161, 162}. The spectral effects are especially interesting near the focal point of a vortex\textsuperscript{125}. A related subject is the investigation of vortices with fractional (non-integer) topological charge\textsuperscript{97}. Experimentally, this type of vortex may be obtained when there is a wavelength mismatch between the “seed” beam and a phase mask, used to embed a vortex into a beam, i.e. the phase shift around the core, produced by the mask for a beam wavelength is not an integer multiple of \( 2\pi \textsuperscript{163, 164} \).

Unlike the spatial coherence between two points in space at the same instance of time, spectral effects may be considered as a manifestation of **temporal coherence** effects, described by the autocorrelation \( \Gamma(\tau) = \Gamma(\vec{r}_i, \vec{r}_1, \tau) \) “along” the direction of propagation. Recently, OV’s were investigated in the temporal coherence domain\textsuperscript{165}. Interference
measurements using a polychromatic partially coherent light source proved the existence of temporal correlation vortices.

2.9. Interaction of Optical Vortices with non-linear crystals

In the last decade, fundamental studies of angular momentum have led to theoretical and experimental investigations of conservation laws in parametric frequency-conversion. Each photon of a vortex beam of topological charge $m$ carries $m$ quanta of orbital angular momentum (OAM)$^{23}$. Mechanical effects due to angular momentum of a vortex have been used to trap and move small particles and atoms$^{13, 14, 60, 166-170}$. Therefore an optical vortex presents an excellent tool for the investigation of the transformation of OAM in 3- and 4-photon processes$^{121, 171}$. From an application point of view, OAM transformation in frequency conversion offers a possibility of optical computing (arithmetical operation).

The process of combining two photons into one photon is known as up-conversion or sum-frequency generation (SFG). The doubling of topological charge in the degenerate sum-frequency generation, known as second-harmonic generation (SHG), was demonstrated first by Dholakia et al$^{172}$ and quickly extended to a non-degenerate SFG$^{173-175}$.

The inverse phenomenon, known as down-conversion, occurs when a photon splits into two photons. Down-conversion may be separated into two fundamentally different cases. One is the optical parametric amplification (OPA), where a seed beam is provided. OPA may be described by coupled-wave theory equations, similar to sum-frequency generation. Conservation of OAM in this case was confirmed by Berzanskis et
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However the process of spontaneous down-conversion (DC), where the pump photon splits into signal and idler photons spontaneously and the role of a seed beam is played by vacuum noise, is not immediately obvious and requires quantum treatment. The theory of DC was developed in the 1970-s\textsuperscript{121}, yet the vortex applications were only treated recently\textsuperscript{177, 178}. Confirmation of OAM conservation in spontaneous down-conversion was obtained by Mair et al.\textsuperscript{179} using photon-correlation counting technique.
**Fig. 2.1:** Single optical vortex of charge +1. (a) Intensity profile, (b) Phase map, (c) – Interferogram with a weak tilted planar reference wave (10% of composite beam amplitude). Interferograms are used to determine the value of vortex charge.
Fig. 2.2: Illustration of the notation relating to the propagation of mutual coherence function.
3. DETECTION OF OPTICAL VORTICES
IN PARTIALLY COHERENT LIGHT

3.1 Introduction

The problem of detection of optical singularities in the coherent light has been addressed using interferometric, mode converter, and holography techniques. The most common way is the analysis of interference fringes, as described in Ch. 2. In practice, the rough position of a vortex in a beam may be verified by locating the vertex of these forks. The precise location of a singularity is then found by removing the reference wave and determining the location of minimum intensity near the vertex. However, interferometry of low coherence light may not provide sufficient fringe visibility for the forking analysis, resulting in the claims that vortices in partially coherent light do not exist.

An alternative method designed to investigate vorticity (helicity of a wavefront) in a beam considers propagation of a beam, truncated by a non-transparent screen (see Ref. and p. 55). When a screen cuts half of the beam, we may call the resulting beam a half-cut or half beam. In the far-field, a half-cut beam is expected to rotate by 90° due to the behavior of the Pointing vector of a vortex beam. In this Chapter, we apply the far-field analysis to half-cut beams in the domain of partial coherence. We measure the average transverse azimuthal component of the \( k \)-vector of a vortex beam. In the far field...
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(k-space) a truncated vortex beam forms a pattern, which has the center of mass of its intensity distribution displaced from the optical axis. We show how the value of this displacement \( k_{cm} \) is related to an average azimuthal \( k \)-vector component. Asymmetry due to the azimuthal component may be expected to persist to the partially coherent case. On the other hand, using a Gaussian beam as an example, we show that a half beam without an azimuthal \( k \)-vector component always has the intensity center of mass on the optical axis. We demonstrate both numerically and experimentally that for a partially coherent half vortex beam the center of mass of an intensity distribution in the far field is displaced from the optical axis. We find a coherence limit, where the beam may be considered as having no azimuthal \( k \)-vector component and the associated vorticity.

The mathematical description of a truncated vortex beam and beam center of mass calculations are given in Sec. 3.2 & 3.3, followed by partial coherence description in Sec. 3.4. In Sec. 3.5 we present a geometrical model of the far-field distribution of a truncated vortex. Sec. 3.6 describes the experimental setup and results and conclusions are in Sec. 3.7.

3.2 Coherent truncated beam

In this chapter we consider a large-core on-axis vortex, placed on a Gaussian beam. In the initial transverse plane \( z=0 \), at time \( t=0 \), and assuming initial phase \( \beta = 0 \), Eq. (2.1) for the vortex transverse profile becomes:

\[
E(r, \phi) = E_0 |r / w_0|^m \exp(-r^2 / w_0^2) \exp(im\phi)
\]

(3.1)
3. DETECTION OF OPTICAL VORTICES IN PARTIALLY COHERENT LIGHT

where polar coordinates \((r, \phi)\) are measured with respect to the center of a beam. The intensity and the phase profile of a field, described by Eq. (3.1) for \(m = 1\), are shown in Fig. 3.1. A truncated beam (for a description of truncated Gaussian beams see \(^{182}\)) does not have radial symmetry, and its field in plane \(z=0\) is more conveniently expressed in terms of Cartesian coordinates \((x, y)\):

\[
E_H(x, y) = \begin{cases} 
E_0 \left(\frac{x + iy}{w_0}\right)^m \exp\left(-\frac{x^2 + y^2}{w_0^2}\right), x \geq s \\
0, \quad \text{otherwise}
\end{cases}
\]  

(3.2)

where we assume truncation by a solid non-transmitting half-plane \((x < s)\). The intensity distribution is given by:

\[
I_H(x, y) = \begin{cases} 
E_0 \left(\frac{x^2 + y^2}{w_0^2}\right)^m \exp\left(-2\frac{x^2 + y^2}{w_0^2}\right), x \geq s \\
0, \quad \text{otherwise}
\end{cases}
\]  

(3.3)

The intensity profiles of a half-Gaussian beam \((m = 0)\) and a half-vortex of charge ‘+1’ \((m = 1)\), obtained by truncating the corresponding beams with a plane \((x < 0)\), are shown on Fig. 3.2(a) and 3.2(b) respectively. We note that the phase transition from 0 to \(2\pi\) in a vortex beam occurs at \(\phi=0\) (see Fig. 3.2(c)).

The field distribution in \(k\)-space is given by the Fourier transform of a field in the plane \((x, y)\):

\[
E^\circ(k_x, k_y) = \frac{1}{w_0^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x, y) \exp(-ik_x x - ik_y y) dx dy
\]  

(3.4)

where \(k_x\) and \(k_y\) denote coordinates in the far-field (\(k\)-space). In order to simplify notation we introduce dimensionless \(k\)-space coordinates \(p_x = k_x w_0, p_y = k_y w_0\). For a Gaussian \((m=0)\) and vortex \((m=1)\) fields we respectively obtain (see Appendix A):
3. DETECTION OF OPTICAL VORTEXES IN PARTIALLY COHERENT LIGHT

\[ E^\infty_{m=0}(p_x, p_y) = E_0 \pi \exp(-p^2/4) \quad (3.5) \]

\[ E^\infty_{m=1}(p_x, p_y) = E_0 (-i \pi / 2) (p_x + i p_y) \exp(-p^2/4) \quad (3.6) \]

where \( p^2 = p_x^2 + p_y^2 \). For a half-beam, described by Eq. (3.2), the far-field distributions for \((m=0)\) and \((m=1)\) fields may be found by integrating a non-truncated field over the right half-plane (see Appendix A):

\[ E^\infty_{H, m=0}(p_x, p_y) = E_0 (\pi / 2) \text{Erfc}(ip_x / 2) \exp(-p^2/4) \quad (3.7) \]

\[ E^\infty_{H, m=1}(p_x, p_y) = E_0 (-i \pi / 4)[(p_x + ip_y) \text{Erfc}(ip_x / 2) + i \exp(p_x^2/2)] \exp(-p^2/4) \quad (3.8) \]

where \( \text{Erfc}(z) \) is a complementary error function\(^\text{183}\). We note that the half-vortex field in \( k \)-space contains a vortex with the center at \( p_x = 0, \ p_y = -1 \). The corresponding intensity profiles \( I(p_x, p_y) = |E(p_x, p_y)|^2 \), calculated from Eq. (3.7) and (3.8) are:

\[ I^\infty_{H, m=0}(p_x, p_y) = E_0^2 (\pi^2 / 4) \text{Erfc}^2(ip_x / 2) \exp(-p^2/2) \quad (3.9) \]

\[ I^\infty_{H, m=1}(p_x, p_y) = E_0^2 (\pi^2 / 16)[p_x^2[1 + \text{Erfi}^2(p_x / 2)] + \exp(p_x^2/2) \]

\[ -2 \exp(p_x^2/4)[p_x \text{Erfi}(p_x / 2 - p_y)] \exp(-p^2/2) \quad (3.10) \]

where \( \text{Erfi}(z) \) is an imaginary error function\(^\text{183}\).

Intensity profiles of a half-Gaussian \((m=0)\) and a half-vortex \((m=1)\) fields, calculated using Eq. (3.9) and (3.10) are shown on Fig. 3.2(d) and (e), respectively. We use \( k_x \) and \( k_y \) coordinates in all figures for clarity. The half-Gaussian beam profile has an elliptical shape, while half-vortex distribution resembles one lobe of a TEM01 mode. We note that unlike that of a Gaussian beam, the intensity profile of a truncated vortex \((m=1)\) is only symmetric with respect to the origin in horizontal direction and is displaced in
vertical direction. The location of a maximum of a half-vortex beam in k-space may be determined by calculating partial derivatives of Eq. (3.10) with respect to $p_y$ for $p_x = 0$:

$$I_{H,m=1}^y(0,p_y) = E_0^2(\pi^2/16)(p_y + 1)^2 \exp(-p_y^2/2) \quad \text{and} \quad \partial I_{H,m=1}^y(0,p_y)/\partial p_y = 0$$

when $[2 - p_y (p_y + 1)] = 0$. Solution of the last equation are $p_y = 1$ and $p_y = -2$ with the maximum of intensity being at $p_x = 0, p_y = 1$ ($k_x = 0, k_y = 1/\omega_0$). We note that a half-vortex distribution in k-space (see phase profile in Fig. 3.2(f)) contains a singularity (vortex) with the center at $p_x = 0, p_y = -1$ ($k_x = 0, k_y = -1/\omega_0$). Examination of the phase profiles also reveals $\pi/2$ Gouy phase shift, indicated by the line of zero phase going along the positive x-axis in the near field and $k_y$-axis in the far field.

The effect of displaced intensity distribution of a vortex beam may be understood by realizing that for a vortex beam the azimuthal transverse k-vector component is positive in the right half of the beam and negative in the left half. Therefore if the beam is truncated by a vertical plane $x < 0$, then all the negative components are removed. The resulting distribution in k-space exhibits the presence of azimuthal transverse k-vector component by lifting the vertical symmetry.

### 3.3 Center of mass calculations

A convenient way to define the symmetry and quantify the azimuthal transverse k-vector component is the calculation of center of mass (CM) of an intensity distribution. The $k_y$-component of CM in k-space is given by:
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\[ k_{c,y} = \frac{1}{w_0} \left( \iint p_x I(p_x, p_y) dp_x dp_y \right) / \left( \iint I(p_x, p_y) dp_x dp_y \right) \]  

(3.11)

where both integrations are over the \( k \)-space and \( w_0 \) is again the Gaussian beam size in \( r \)-space. If intensity \( I(p_x, p_y) \) is zero outside the beam area \( A \), then integrations may be instead conducted over \( A \), allowing calculation of center of mass on experimental images.

Analysis of Eq. (3.9) reveals that \( I_{H,m=0}^x(p_x, p_y) \) is an even function with respect to both \( p_x \) and \( p_y \): \( I_{H,m=0}^x(p_x,-p_y) = I_{H,m=0}^x(p_x, p_y) \) and \( I_{H,m=0}^x(-p_x, p_y) = I_{H,m=0}^x(p_x, p_y) \). Therefore, the center of mass of its distribution is at the origin \((p_x = 0, p_y = 0)\). The distribution is symmetrical as illustrated in Fig. 3.2(d). Similar results may be obtained for other beams with no azimuthal \( k \)-vector component.

In contrast, the intensity distribution of a vortex beam (see Fig. 3.2(e)) is not symmetric in the vertical direction. For example, the \( k_y \)-component of the center of mass of a \( k \)-space half-vortex \((m=1)\) field, calculated using Eq. (3.10) and numerical integration of Eq. (3.11), is given by:

\[ k_{c,y} \approx 0.71 / w_0 \]  

(3.12)

Eq. (3.11) may be expected to hold in the partially coherence regime, however the intensity distribution is expected to change due to the beam spreading. In the next chapter we numerically calculate intensity distributions and center of mass coordinates for a partially coherent beam, whose envelope is given by Eq. (3.1) for \( m=0, 1 \).
3. DETECTION OF OPTICAL VORTICES IN PARTIALLY COHERENT LIGHT

3.4 Partially coherent half-beam

A partially coherent beam may be characterized by the mutual coherence function (MCF), introduced in the previous chapter and described by Eq. (2.2). A vortex, described by Eq. (3.2), may be obtained by passing a beam through a spiral phase-amplitude mask. A beam propagating through a mask may be expected to gain the characteristic vortex phase factor $\exp(im\phi')$ even when its wavefront is randomized. For a thermal source it is then convenient to use Eq. (2.5) form and describe coherence properties of a beam in the mask plane with a Gaussian-Shell correlator, given by Eq. (2.8). For a beam described by Eq. (3.2), the MCF takes the form:

$$\Gamma_{H,m}(\vec{r}_1, \vec{r}_2) = \begin{cases} 
E_0[(x_1 + iy_1)(x_2 - iy_2)/w_0^2]m \exp(-[r_1^2 + r_2^2]/w_0^2) \\
\exp(-[\vec{r}_1 - \vec{r}_2]^2/I_c^2), \ x \geq s \\
0, \ otherwise 
\end{cases} \quad (3.13)$$

The intensity distribution in the near-field is given by $I_{H,m}(\vec{r}) = \Gamma_{H,m}(\vec{r}, \vec{r})$, see Eq. 2.3, and takes the form:

$$I_{H,m}(\vec{r}) = \begin{cases} 
E_0[r/w_0]^2m \exp(-2r^2/w_0^2), \ x \geq s \\
0, \ otherwise. 
\end{cases} \quad (3.14)$$

i.e. is the same as that for a coherent beam, see Eq. 3.3. The independence of the coherence length in the near-field is to be expected in a model based on Eq. (2.5) and (2.8), which describes properties of a light source. Coherence properties of a beam require non-zero propagation distance to affect the field and intensity distribution.
Here we are interested in the far-field distribution. In order to find the intensity distribution of a partially coherent beam in $k$-space, one may calculate the far-field distribution of the mutual coherence function and then use the relation

$$I^\infty_{H,m}(\vec{k}) = \Gamma^\infty_{H,m}(\vec{k},\vec{k})$$

where $\vec{k}$ is the $k$-vector. The far-field distribution for the MCF of a half-beam, given by Eq. (3.2), is given by the integral transform:

$$\Gamma^\infty_{H,m}(\vec{k}_1,\vec{k}_2) = \iint \Gamma_{H,m}(\vec{r}_1,\vec{r}_2) \exp[-i(\vec{k}_1 \cdot \vec{r}_1 - \vec{k}_2 \cdot \vec{r}_2)] d\vec{r}_1 d\vec{r}_2$$  \hspace{1cm} (3.15)

which allows the separation of variables and a numerical solution. The intensity distribution for a vortex and a Gaussian beam are respectively given by:

$$I^\infty_{m=0}(\vec{k}) = E_0^2 (\iint T_x dx_1 dx_2 \iint T_y dy_1 dy_2)$$ \hspace{1cm} (3.16)

and

$$I^\infty_{m=1}(\vec{k}) = (E_0^2 / w_0^2)(\iint x_1 x_2 T_x dx_1 dx_2 \iint y_1 T_y dy_1 dy_2 + \iint x_1 T_x dx_1 dx_2 \iint y_2 T_y dy_1 dy_2 + \iint T_x dx_1 dx_2 \iint y_2 T_y dy_1 dy_2)$$ \hspace{1cm} (3.17)

where

$$T_x = \exp(-a(x_1^2 + x_2^2) + 2bx_1x_2) \cdot \exp[ik_x \cdot (x_2 - x_1)]$$

$$T_y = \exp(-a(y_1^2 + y_2^2) + 2by_1y_2) \cdot \exp[ik_y \cdot (y_2 - y_1)]$$ \hspace{1cm} (3.18)

and $(x_1, y_1), (k_x, k_y)$ are the coordinates of $\vec{r}_1$ and $\vec{k}$, $a = 1/w_0^2 + 1/l_C^2$ and $b = 1/l_C^2$.

Numerically calculated intensity distributions for a full and half-cut vortex beams and $l_C/w_0 = 0.1; 2.0$ are shown in Fig. 3.3. The intensity profile of a full beam for $l_C/w_0 = 0.1$, shown in Fig. 3.3(a), reveals that the core of a low-coherent vortex beam is filled with diffused light. In contrast, Fig. 3.3(b) demonstrates an increase in the core visibility with an increase in coherence length.
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For a highly incoherent ($l_C = 0.1w_0$) half-vortex beam the $k$-space distribution appear to be symmetric, as shown on Fig. 3.3(c). However, this symmetry lifts as the coherence length increases (see Fig. 3.3(d)), indicating the presence of a transverse azimuthal $k$-vector component. Nevertheless the maximum of intensity distribution appear to be a constant, independent of the coherence length. For a given field of view the transverse size of the distribution grows as we decrease the coherence length. For a highly incoherent beam the field of view may be completely filled with light, disallowing accurate measurements of the center of symmetry of intensity distribution. We shall compare calculated locations of the center of symmetry of intensity distribution with experimental results in Sec. 3.6.

3.5 Geometrical model

An intensity distribution of a partially coherent beam may be illustrated by the schematic diagram shown on Fig. (3.4). We consider propagation of a beam with a plane wavefront from the plane ($x$, $y$) and neglect the radial $k$-vector components. Following Huygens principle, each microscopic area of a beam radiates a cone of light, defined by the beam spread, which in turn is defined by the coherence length and the radiating area size (17, also see Ch. 2).

The conical projections define circles in the plane ($x'$, $y'$). The distance $z$ between planes ($x$, $y$) and ($x'$, $y'$) is assumed to be greater than the far-field distance (Fraunhofer regime, see 17). For a coherent beam with a planar wavefront the circle size is defined by the radiating area size, which we assume to be sufficiently large so that diffraction effects
may be neglected and the spread of a cone of light, coming from the area, is only defined by the coherence properties. The location of a circle in the plane \((x', y')\) is then defined by the transverse \(k\)-vector component \(k_T\) of a field. For convenience, we call the center of a circle a point \(C\), and the normal projection of the radiating point into plane \((x', y')\) a point \(P\). If \(k\)-vector \(\vec{k}\) is normal to the plane \((x, y)\), then points \(P\) and \(C\) would coincide. However, if \(k_T\) is not zero, then the distance between points \(P\) and \(C\) would be proportional to \(k_T\) [and to the distance between planes \((x, y)\) and \((x', y')\)] (see Fig. 3.4, cones with an origin at \((r, 0)\)).

We may now consider the same cone-model in \(k\)-space (see Fig. 3.5). All radiating points are now at the origin and the cones are defined by \(k_z\) (height) and \(k_T\) (center location). The size of a cone base is given by Eq. (2.8): \(R_{\text{CW}} / z \cong (2\pi)^{1/2} / k l_c\).

The projections of cone bases onto the transverse plane \((k_x, k_y)\) represent a 2-dimensional transverse \(k\)-space distribution. We note that although in general a point in the \(x\)-space illuminates all points in the \(k\)-space, in our model we neglect the light outside the cone.

We note that the point ‘0’, which had a positive vertical \(k_T\) in \(x\)-space (Fig.3.4), is now represented by a cone with a center displaced in the positive vertical direction (see Fig.3.5(a)). Similar relations exist for other cones. Compared to \(x\)-space, the vortex beam in \(k\)-space is rotated by an angle \(\pi/2\), with each point from \(x\)-space transforming onto a circle of a radius defined by the coherence length.

We may now consider the \(k\)-space distribution of a half-vortex field, as shown in Fig. 3.5(b). For a coherent beam, the beam spread is negligible and the far-field, composed of cones coming from all \(x\)-space points of a beam, has an asymmetric
distribution. As the beam becomes less coherent, the beam spread and the base sizes of the cones grow. For a beam with a very low value of coherence length, the cone base size is comparable with the beam size and may actually exceed it (termed highly incoherent light). Under these conditions, we may expect the $k$-space distribution of a partially coherent half-vortex beam to become symmetric. We note that the deviation from symmetry was due to the transverse azimuthal component of the $k$-vector. Therefore, we may use the transition from asymmetric to symmetric $k$-space distribution as the criteria of the presence of azimuthal $k$-vector component in a beam. The peculiar features of a vortex beam are due to this component and therefore the same transition may serve as the criteria for the existence of a vortex in a partially coherent beam.

### 3.6 Half-vortex experiment

Experimentally, the image of a $k$-space distribution may be obtained by using a lens as a Fourier-transforming system. The main challenge here is the imaging near the focus. We solve this problem by focusing a partially coherent beam on a vortex phase mask, placed in the focus of a Fourier-transforming lens. The experimental setup is shown on Fig. 3.6. Broadband light from an incoherent halogen source passes through two apertures. The spatial coherence is controlled by varying the aperture $Ap_1$ of radial size $A$ (set of apertures 5; 15; 25; 50; 100 $\mu m$; precision of $\pm 1\mu m$), while the second aperture $Ap_2$ has a radial size $w=5 \pm 0.5mm$. The distance between the apertures is $B=75\pm1mm$. The transverse coherence width $l_C$ at the second aperture is approximately given by
$l_c = 0.64 \overline{\lambda} B / A^{17}$, where $\overline{\lambda}$ is the average wavelength. Aperture $A_{p2}$ is imaged with unity magnification by the lens $L_1$ of focal length $f_1 = 75.6 \text{mm}$ into the front focal plane of lens $L_2$ of focal length $f_2 = 125 \text{mm}$. The vortex phase mask (PM), designed to produce an $m=1$ vortex at wavelength $\lambda_0 = 890 \text{nm}$ and air interface, is placed in the back focal plane of lens $L_2$. Our mask had 8 steps of different thickness, given by $d = d_0 + \lambda_0 n / 4 (n_s - n_0)$, where $d_0 \cong 0.5 \text{mm}$ is the minimum thickness, $n$ is the step number [steps are counted in counterclockwise direction], and $n_s$ and $n_0$ are indices of refraction of the substrate and of the surrounding medium. Therefore, each step differs from an adjacent neighbor in the introduced phase delay of $\pi/4$. The phase mask position in all 3-dimensions is controlled with a micrometer, allowing in- and out-of focus, as well as on- and off-axis center location with a precision of $1 \mu\text{m}$. The mask plane is imaged by a lens $L_3$ of focal length $f_3 = 50 \text{mm}$ with a unity magnification into a razor plane $z=z_r$. The razor is placed on a micrometer stage, allowing truncation of a beam in the transverse plane with a precision of $1 \mu\text{m}$. The razor plane is Fourier-transformed with a lens $L_4$ ($f_4=200 \text{mm}$) onto a cooled CCD camera (Mead model 416XT). A 50nm band pass filter with a mean transmitted wavelength of $\overline{\lambda} = 800 \text{nm}$ is attached to the front of the camera to achieve quasi-monochromaticity and minimize temporal coherence effects.

Typical experimental intensity distributions in the far-field are shown on Fig. 3.7 (Fig. 3.7(a), (b) show a non-truncated beam, and Fig. 3.7(c),(d) show a half-beam, truncated by the plane $x < 0$). For a beam in Fig. 3.7(a), (c) the aperture size is $a = 15 \mu\text{m}$. For $\overline{\lambda} = 800 \text{nm}$, the coherence length at the aperture $A_{p2}$ is thus given by $l_c = 2.56 \text{mm}$. The
coherence length relative to the beam size is \( l_c / w = 2.56(mm) / 5(mm) \approx 0.5 \). The visibility of the vortex core is calculated using the raw images (not shown). The output of the CCD camera is measured on the digital scale 0 to 65535. The intensity values are calculated with respect to the maximum output of 65535. Maximum intensity \( I_{\text{max}} = 0.664 \) and minimum (located inside the core) \( I_{\text{min}} = 0.0625 \). Therefore, the visibility \( \eta = (I_{\text{max}} - I_{\text{min}}) / (I_{\text{max}} + I_{\text{min}}) \approx 0.83 \). For a highly incoherent beam (Fig. 3.7(b),(d)), the aperture size is \( a = 50 \mu m \), giving \( l_c = 0.77 mm \). The relative coherence length is \( l_c / w \approx 0.15 \) and the core visibility is \( \eta \approx 0.53 \). For larger aperture sizes, core visibility drops below the noise level (determined by the intensity noise, noise visibility is approximately 0.05).

Fig. 3.8 shows \( y \)-coordinate of \( k \)-space intensity center of mass. This coordinate is calculated using experimental images and is shown relative to the coherent case (maximum) value, \( y_c \). For comparison purposes we also show center of mass \( y \)-coordinate, obtained using analytically calculated distribution of half-Laguerre-Gaussian vortex (see Eq. (3.10)), as well as visibility of experimental images. Both coordinates and visibility are shown as functions of the relative coherence length. The center of mass coordinates are calculated using Eq. (3.11) on a raw intensity image. The location of the center of mass of an analytically calculated distribution in coherent case (maximum value) is given by Eq. (3.12). We note that the intensity distribution becomes symmetric, \( y \to 0 \), and the core visibility approaches zero \( (\eta \to 0) \) for highly incoherent beams \( (l_c / w < 0.1) \). Experimental and numerically calculated values are found to be in
excellent agreement. We note however that our calculations of the center of mass have qualitative nature because the beam size (see Fig. 3.7) generally exceeds the field of view. One may expect that an increased field of view may allow detection of the center of mass displacement for even lower values of coherence length. We note that in our numerical model we assume a Gaussian beam, while in the experiment the source has a spherical wavefront cut by an aperture.

3.7 Conclusion

Our investigation indicates that the helicity of optical vortices in partially coherent light may be detected using $k$-space analysis of truncated beams. Our technique allows direct quantification of helicity by measuring the asymmetry of the $k$-space intensity distribution for a truncated beam. Accuracy of our technique is limited in the case of low coherence lengths, when far field distribution size may exceed the camera field of view. Compared to other techniques, designed to investigate the vortex phase structure, our method does not require a reference beam.
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Fig. 3.1: Vortex intensity and phase
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Fig. 3.2: Distributions of coherent truncated beams. (a), (b), (c) show the x-space distribution of half-Gaussian beam (truncated LG mode, m=0) intensity, and half-vortex (truncated LG mode, m=1) intensity and phase, respectively. (d), (e), (f) show corresponding distributions [(a)-(d), (b)-(e), (c)-(f)] in k-space.
Fig. 3.3: Numerically calculated k-space intensity distributions of partially coherent beams. (a, b) show full and (c, d) show half-cut vortex beams for relative coherence length $l_c/w_0 = 0.1$ (a, c), and $2.0$ (b, d).
Fig. 3.4: A diagram of partially coherent vortex beam propagating from the xy-plane. Light from a point in xy-plane forms a cone due to beam spread. In the plane x’y’ a conical projection forms a circular region. We show 5 such circles, produced by points with polar coordinates (0,0), (r, 0), (r, π/2), (r, π), (r, 3π/2) in xy-plane. Vortex beam k-vectors at all but central point have transverse components, which result in the displacement of x’y’-plane projections in a azimuthal direction.
Fig. 3.5: A diagram of beam-spread conical projections in k-space. (a) Cones $0$, $\pi/2$, $\pi$, $3\pi/2$ originate from the points (0,0), (r, 0), (r, $\pi/2$), (r, $\pi$), (r, $3\pi/2$) in x-space, as shown in Fig. 3.4. As a result of a presence of transverse azimuthal k-vector components, the projections of cones into k$_x$k$_y$ plane are displaced from the origin and appear to be rotated by 90° around the beam center from their x-space location. (b) The beam is truncated by x<0 plane in x-space. The light from point $\pi$ is now blocked, creating k-space distribution, which is asymmetrical in vertical plane.
An experimental setup shows light from a spatially incoherent light source \textit{ILS} (halogen bulb) with a mean wavelength of 800 nm and a bandwidth of 50 nm passing through two apertures separated by a distance $B = 7.5 \text{cm}$. The $Ap_2$ plane is imaged with unity magnification by a lens $L_1$ (focal length $f_1 = 75.6 \text{mm}$) into a front focal plane of lens $L_2$ (focal length $f_2 = 125 \text{mm}$). A phase mask $PM$, designed to produce an $m=1$ vortex at wavelength $\lambda = 890 \text{nm}$ and air interface, is placed in a second focal plane of $L_2$. The mask plane is imaged with a unity magnification by lens $L_3$ (focal lengths $f_3 = 50 \text{mm}$) into a razor plane $z_r$. Plane $z_r$ is in the front focal plane of lens $L_4$ ($f_4 = 200 \text{mm}$). A cooled CCD camera (Mead model 416XT) is placed into back focal plane of $L_4$. 

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Fig. 3.7: Experimental images. (a), (b) show vortices the far-field distribution of a non-truncated vortex beam for the relative coherence length $l_c/w=0.5, 0.15$. The core visibility decreases with the decrease in coherence. (c), (d) show far-field distributions obtained respectively with beams from (a) and (b), truncated in the near-field by a razor, which blocks left half of a beam. Both distributions contain dark areas, displaced in vertical direction.
Fig. 3.8: Center-off-mass $y$-coordinate and core visibility vs. coherence length.
4. COMPOSITE OPTICAL VORICES

4.1 Introduction

When one wave is superimposed with another, the phase of the resulting composite field may differ from that of the composing waves. Being a topological phase structure, an optical vortex is readily affected when it is combined with other fields. The new or “composite vortex” may almost instantly appear to be repositioned in space or annihilated, and other vortices may spontaneously form in the net field. The velocity at which a vortex actually moves depends on how quickly parameters such as the relative phase or amplitude can be varied. For example, the rate of change of the amplitude can be so large for an ultra-short pulse of light that the vortex speed exceeds the speed of light. This finding does not violate the laws of relativity because the vortex velocity is a phase velocity.

Thus far the study of beam combinations has been limited to the coherent co-axial superposition of several beams. For example Soskin and Vasnetsov showed that a coherent background field changes the position of an OV and can lead to the destruction and creation of vortices. In this Chapter we describe OV’s in the superimposed field of two parallel, non-collinear coherent beams. This lays the groundwork for the introduction of correlation vortices in Ch. 5. Description of a single vortex is given in Sec. 2. In Sec. 3 we map the position and topological charge of the OV’s as the relative phase or distance between the two beams is varied. For consistency with experimental approaches, we
numerically generate interferograms that allow the determination of the position and topological charge of vortices in the composite field. The potential of composite vortices is illustrated in Sec. 4, where we discuss the velocity of composite vortex motion. Sec. 5 presents the conclusion.

4.2 Single Optical Vortex

For a single on-axis optical vortex placed at the center of a cylindrically symmetrical coherent beam Eq. (2.1) in the transverse \((x, y)\) plane \(z=0\) and at time \(t=0\) may be expressed as:

\[
E(r,\phi) = E_0 A(r) g(r) \exp(i m \phi + i \beta) \quad \text{(4.1)}
\]

where, following notation of Ch. 2 & 3 (see Fig. 3.1), \((r, \phi)\) are circular transverse coordinates in the \((x, y)\) plane, \(E_0\) is a measure of the field amplitude (assumed to be real), \(\beta\) is an arbitrary phase constant, \(g(r)\) is a real function representing the profile of the beam envelope, \(A(r)\) is a vortex core profile, and \(m\) is the topological charge of the vortex (a signed integer). The amplitude and phase may vary with time. In principle any vortex of topological charge, \(m\), may be treated as the product of \(|m|\)-vortices, each having a fundamental charge of \(\pm 1\). Thus, for simplicity we consider only beams with fundamental topological charges, \(m=-1, 0, \text{ or } +1\). Below we shall make use of the relation between the amplitude and power of the beam in Eq. (4.1):

\[
E_0 = \left(\frac{P}{\int_0^\infty r A^2(r) g^2(r) dr}\right)^{1/2} \quad \text{(4.2)}
\]
As mentioned in the introduction, a characteristic of an optical vortex is a point of zero intensity in the dark vortex core. Analytically, the position of this zero may be found by considering that both the real and imaginary parts of the field must vanish:

\[
\text{Re}\{E\} = E_0 A(r) g(r) \cos(m\phi + \beta) = E_0 r^{-1} A(r) g(r) x' \equiv 0 \quad (4.3)
\]

\[
\text{Im}\{E\} = E_0 A(r) g(r) \sin(m\phi + \beta) = E_0 r^{-1} A(r) g(r) y' \equiv 0 \quad (4.4)
\]

where \((x',y') = (r \cos(m\phi + \beta), r \sin(m\phi + \beta))\) are rotated coordinates. Both Eq. (4.3) and (4.4) vanish identically at the origin for any physically meaningful \(A(r)g(r)\) product. In this chapter we restrict ourselves to Gaussian beam profiles: \(g(r) = \exp(-r^2/w_0^2)\), where \(w_0\) is again the characteristic beam size. Here we shall employ Eq. (4.3) and (4.4) to find zero intensity points and then use forking analysis (Ch. 2) to verify vortex presence.

### 4.3 Composite Vortices

When two or more beams described by Eq. (4.1) and possibly having different centroids are superimposed, the resultant zero intensity points in the composite beam do not generally coincide with those of the individual beams. Being phase objects we expect the ‘new’ composite vortices to depend on the relative phase and amplitude of the individual beams. Let us first examine the superposition of two mutually coherent beams containing vortices under steady state conditions. The composite field is given by

\[
E(r, \theta) = \sum_{j=1}^{2} A_j g(r_j)(r_j/w_0)^{im_j} \exp(im_j \phi_j) \exp(i\beta_j) \quad (4.5)
\]

where \((r_j, \phi_j)\) are the transverse coordinates measured with respect to the center of the \(j\)-th beam, and \(\beta_j\) is the phase of the \(j\)-th beam. The constant \(A_j\) represents the relative
amplitude of the \( j \)-th beam and we assume a large-core vortex. The location of the composite vortices may be found by using interference with a planar reference wave \( E' = E_0' \exp(ik_x x) \), as described in Ch. 2. The intensity profile of the interferogram is given by

\[
|E + E' \exp(ik_x x)|^2 = |E|^2 + E_0'^2 + 2|E|E_0' \cos(\Phi - k_x x)
\]

(4.6)

where \( \Phi = \arctan(\text{Im}\{E\}/\text{Re}\{E\}) \). The intensity of the interferogram at the location of a composite vortex has a value given by \( E_0'^2 \) (since \( |E| = 0 \)) and forking patterns are expected around the singular points where \( \Phi \) is undefined.

For convenience we consider two beams displaced along the \( x \)-axis by a distance \( \pm s \) from the origin as shown in Fig. 4.1, where Beam-1 and Beam-2 appear on the right and left, respectively. The relative displacement compared to the beam size is defined by \( \sigma = (s/w_0) \). One may readily verify that the bipolar coordinates \((r_1, \phi_1, r_2, \phi_2)\) are related to the circular coordinates \((r, \phi)\) by the relations

\[
r_1 \exp(\pm i \phi_1) = r \exp(\pm i \phi) - s
\]

(4.7)

\[
r_2 \exp(\pm i \phi_2) = r \exp(\pm i \phi) + s
\]

(4.8)

Without loss of generality, we define the relative phase, \( \beta = \beta_1 - \beta_2 \) and set \( \beta_2 = 0 \). Let us now consider three representative cases of combined beams where the topological charges are identical \((m_1=m_2)\), opposite \((m_1=-m_2)\), and different \((m_1=1, m_2=0)\).

**Case 1:** If we superimpose two singly charged vortex beams having identical charges, say \( m_1 = m_2 = +1 \), then the composite field in Eq. (4.5) may be expressed
\[ E(r, \theta) = [A_0 g_1(r_1) \exp(i\beta) + A_2 g_2(r_2)](r/w_0) \exp(i\phi) \]
\[-[A_0 g_1(r_1) \exp(i\beta) - A_2 g_2(r_2)](s/w_0) \]

The intensity profiles, |\(E|^2\), for various values of phase and separation are shown in Fig. 4.2. Setting the real and imaginary parts of Eq. (4.9) to zero, we find composite vortices located at the points (x, y) that satisfy the transcendental equation

\[ x + iy = s \tanh \left( 2\sigma^2 x/s + (1/2) \ln(A_1/A_2) + i\beta/2 \right) \]

Several special cases of Eq. (4.10) can be readily identified and solved.

Let us first consider the in-phase and equal amplitude case: \(\beta = 0\) and \(A_1 = A_2\), which allows solution only along the x-axis (see the top row in Fig. 3). A solution of Eq. (4.10) exists at the origin for any value of \(\sigma\), however we find its sign changes from positive to negative when \(\sigma\) increases beyond the critical value:

\[ \sigma_{cr} = 2^{-1/2} \]

What is more, for \(\sigma > \sigma_{cr}\), two additional positive vortices emerge. One may easily verify this critical value by expanding the hyperbolical tangent function in Eq. (4.10) to third order, assuming \(|x/s| << 1\), and thereby obtaining

\[ |x/s| \approx \left[ 3(2\sigma^2 - 1)/8\sigma^6 \right]^{1/2} \]

For well separated beams, \(\sigma >> \sigma_{cr}\), a first order expansion of Eq. (4.10) indicates that the composite vortices are displaced toward each other \(|x/s| < 1\), and as expected, they nearly coincide with the location of the original vortices:

\[ |x/s| \approx 1 - 2\exp(-4\sigma^2 x/s) \]
Let us next consider the out-of-phase, equal amplitude case: $\beta = \pi$, $A_1 = A_2$ (see bottom row in Fig. 4.2). In this case we find two vortices for all values of $\sigma$, with an additional vortex at $y = \pm \infty$. For the special case $\sigma = 0$, total destructive interference occurs and the composite field vanishes. Again we find the solutions are constrained to the x-axis but now the positions are given by

$$x/s = \coth[2\sigma^2 x/s]$$

(4.14)

The vortices are displaced away from each other $(|x/s| > 1)$, and for $\sigma >> \sigma_{cr}$

$$|x/s| \cong 1 + 2 \exp(-4\sigma^2 x/s)$$

(4.15)

These analytical results and the numerically calculated intensity profiles in Fig. 4.2 demonstrate that the vortex core may be readily repositioned by changing the relative position or phase of the component beams. From an experimental and application point of view the later variation is often easier to achieve in a controlled linear fashion. For example, the net field may be constructed from two collinear beams from a Mach-Zehnder interferometer, whereby the phase is varied by introducing an optical delay in one arm of the interferometer.

A demonstration of the phase sensitive vortex trajectory is shown in Fig. 4.3 for the case $\sigma = 0.47$ (i.e., $\sigma < \sigma_{cr}$). We find that a critical value of phase exists such that for $|\beta| < \beta_{cr}$ there is a single composite vortex of charge $m = 1$, and it resides on the y-axis. Above the critical phase, two $m = 1$ vortices split off from the y-axis, and simultaneously, one $m = -1$ vortex remains on the y-axis. The value of the topological charge may be found by examining numerically generated phase profiles or interferograms of Eq. (4.9).
The positively charged vortices (see the black dots in Fig. 4.3) circumscribe the position of the original vortices (indicated by isolated gray dots in Fig. 4.3).

The net topological charge is \( m_{\text{net}} = 1 \) for all values of \( \beta \), and, being unequal to the sum of charges of the original beams \( (m_{\text{sum}} = 2) \), is therefore not conserved in the sense that \( m_{\text{net}} \neq m_{\text{sum}} \). From a practical point of view, however, we note that the \( m = -1 \) vortex escapes toward \( |y| = \infty \) when \( \beta \approx \pi \). Furthermore the field amplitude surrounding the escaping vortex is negligible, and is therefore unobservable. Empirically this leads one to the observation that the net topological charge is sometimes conserved in a “weak sense”.

Phase-dependent vortex trajectories for different values of the separation parameter, \( \sigma \), are shown in Fig. 4.3(B). For \( \sigma < \sigma_c \) a single \( m = 1 \) vortex resides on the \( y \)-axis when \( |\beta| < \beta_c \); above the critical phase an \( m = -1 \) vortex continues moving on the \( y \)-axis toward infinity while two \( m = 1 \) vortices form symmetric paths that circumscribe the positions of the original vortices. For \( \sigma > \sigma_c \) three vortices always exist, with one having charge \( m = -1 \) constrained to the \( y \)-axis, and the other two having charge \( m = 1 \) circling the original vortices. As discussed above for Fig. 4.3(A), the net topological charge is never conserved for the cases in Fig. 4.3(B).

The value of \( \beta_{cr} \) may be estimated from Eq. (4.10) by expanding the hyperbolical tangent function in the vicinity of \( x/w_0 = 0 \). To first order we find

\[
2\sigma^2(1 + \tan^2(\beta_{cr}/2)) \approx 1
\]

(4.16)

For \( \sigma = 0.50 \) we obtain \( \beta_{cr} \approx 90^\circ \), in good agreement with the numerically obtained value.

A qualitative understanding of the composite vortex generation can be formed by examining Fig. 4.2. When \( \sigma = 0 \) we have the simple interference between beams having
different phases. When $\sigma=0.63$ we find an elongated composite beam having a single vortex when $\beta=0$, but the emergence of two $m=1$ and one $m=-1$ vortices when $\beta_{cr}<\beta<\pi$.

**Case 2:** For two oppositely charged vortices ($m_1=-m_2=+1$) Eq. (4.5) becomes:

$$E(r,\theta) = A_1 g_1(r_1)(r/w_0)\exp(i\beta)\exp(i\phi) + A_2 g_2(r_2)(r/w_0)\exp(-i\phi)$$

$$- [A_1 g_1(r_1)\exp(i\beta) - A_2 g_2(r_2)](s/w_0)$$

(4.17)

The intensity profiles for different values of phase and separation, shown in Fig. 4.4, are qualitatively different than those for Case-1 (see Fig. 4.2). To determine the position of vortices we set Eq. (4.17) to zero. It happens that the zero-valued field points for both the in-phase and out-of-phase cases, $\beta = 0$ and $\beta = \pi$, must satisfy the equations:

$$x = s \tanh[2\sigma^2x/s + (\sqrt{2})\ln(A_1/A_2) + i\beta/2]$$

(4.18)

$$y \sinh[2\sigma^2x/s + (\sqrt{2})\ln(A_1/A_2) + i\beta/2] = 0$$

(4.19)

For the in-phase equal amplitude case ($\beta=0$, $A_1=A_2$) there are zeros that do not correspond to a vortex, but rather to an edge dislocation (i.e., a division between two regions having a phase difference of $\pi$). The first row in Fig. 4.4 shows, for different values of $\sigma$, the dislocation as a black line bisecting two halves of the composite beam. Vortices do not appear in this in-phase case unless $\sigma > \sigma_{cr}$ [see Eq. (4.11)], whence they occur on the x-axis and are described by the same limiting relations as we found in Case 1 [i.e., Eqs. 4.12 and 4.13]. The out-of-phase equal amplitude ($\beta=\pi$, $A_1=A_2$) solutions are also identical to those found in Case 1 [see Eq. (4.14)].

The composite vortex positions in Case 2 are generally different than Case 1 for arbitrary values of $\beta$ and $\sigma$. For example, in Case 1 we found either one or three
vortices, while in Case 2 there are always two vortices. Whereas the vortex positions in Case 1 appeared symmetrically across the y-axis for $|\beta|>\beta_{cr}$, they appear at radially symmetric positions in Case 2 for all values of $\beta$. Finally we note that when $\sigma=0$ in Case 2 the two original vortices interfere for all values of $\beta$ to produce an edge dislocation, as shown in the left hand column of Fig. 4.4, whereas in Case 1 the beams undergo uniform destructive interference.

The phase-dependent vortex trajectory for $\sigma=0.47$ is shown in Fig. 4.5(A). Indeed it differs from the trajectory for Case 1 shown in Fig. 4.3(A). At $\beta=\varepsilon$ (where $\varepsilon<<1$) we find in Fig. 4.5(A) an $m=1$ vortex at a point on the positive y-axis, and an $m=-1$ vortex at the symmetrical point on the negative y-axis. As the phase advances, the vortex positions rotate clockwise. Curiously we find that as the phase is varied slightly from $\beta=-\varepsilon$ to $\beta=+\varepsilon$ the vortices suddenly switch signs. This switch may also be interpreted as an exchange of the vortex positions. In either case, this exchange occurs via an edge dislocation at the phase $\beta=0$. Edge dislocations are often associated with sources or sinks of vortices. Unlike Case 1, we find that the net topological charge conserved for all values of $\beta$, i.e., $m_{net}=m_{sum}$.

Generic shapes of the composite vortex phase-dependent trajectories are shown in Fig. 4.5(B) for different separation distances. For small values of separation ($\sigma<<1$) the path of each vortex is nearly semi-circular vortices on opposing sides of the origin. When $\sigma>\sigma_{cr}$ each vortex trajectory forms a closed path. Regardless of the value of $\sigma$, the vortex on the right has a charge, $m=1$, opposite of that of the left. As found in Case 1, the critical separation distance delineates open-path and closed-path trajectories. The
topological charge is conserved for all the values of $\beta$ and $\sigma$ in Fig. 4.5, and thus, by induction, we conclude that the net topological charge is always conserved for Case 2.

**Case 3:** Lastly we consider the interference between beams having a vortex (Beam-1) and planar (Beam-2) phase: $m_1 = 1, m_2 = 0$. Examples of the intensity profiles are shown in Fig. 4.6. Substitution of the values of $m$ into Eq. (4.5) gives

$$E(r, \theta) = \mathcal{A}_1 g_1(r_1) \exp(i\beta)(r/w_0) \exp(i\phi) + \mathcal{A}_2 g_2(r_2) - \mathcal{A}_1 g_1(r_1) \exp(i\beta)(s/w_0) \quad (4.20)$$

Zero field points must satisfy:

$$\left(x + iy\right)/s = 1 - \sigma^{-1} \left(\mathcal{A}_2/\mathcal{A}_1\right) \exp(-i\beta) \exp(-4\sigma^2 x/s) \quad (4.21)$$

Let us consider the case when the vortex and gaussian beams have the same power. From Eq. (4.2) we obtain the relation between the amplitudes of the vortex, $\mathcal{A}_1$, and Gaussian, $\mathcal{A}_2$, beams:

$$\mathcal{A}_1 = 2^{\mathcal{V}} \mathcal{A}_2 \quad (4.22)$$

In the in-phase case ($\beta = 0$) Eq. (4.21) simplifies to

$$\left(x + iy\right)/s = 1 - \sigma^{-1} 2^{-\mathcal{V}} \exp(-4\sigma^2 x/s) \quad (4.23)$$

which has solutions only along the x-axis, and two critical points $\sigma_1 \approx 0.1408$ and $\sigma_2 \approx 0.6271$ which are the solutions of the transcendental equation

$$2^{\mathcal{V}} \sigma = \exp(4\sigma^2 - 1) \quad (4.24)$$

The in-phase case has no solutions if $\sigma_1 < \sigma < \sigma_2$ (although a dark spot may be present), one solution if $\sigma = \sigma_1$ ($x/s \approx -11.6$) or $\sigma = \sigma_2$ ($x/s \approx 0.364$) and two solutions otherwise.
Two other cases of special interest exist. An exact solution of Eq. (4.21) is found for any value of $\sigma$ when $\beta = \pi/2$:

$$x = s$$

$$y = (w_o/2^{1/2}) \exp(-4s^2/w_o^2)$$

(4.25) (4.26)

For the out-of-phase case, $\beta = \pi$, Eq. (4.21) simplifies to

$$(x + iy)/s = 1 + \sigma^{-1}2^{-1/2} \exp(-4\sigma^2 x/s)$$

(4.27)

which allows a single composite vortex solution on the $x$-axis whose position is displaced away from the Gaussian ($m=0$) beam.

Numerically determined vortex positions are shown in Fig. 4.7 for $\sigma=0.47$. Since this value falls between the two critical values, there are no composite vortices for the in-phase case. In fact we find no vortices over the range $|\beta|<\beta_{cr}$. The critical phase value depends on the value of $\sigma$, and in this case $\beta_{cr} \approx 40^\circ$. In general the critical phase may be computed from the relation $2^{3/2}\sigma \cos \beta_{cr} \exp(4\sigma^2-1)=1$. The dashed line in Fig. 4.7(A) indicates the position of a dark (but non-zero) intensity minimum that appears when $|\beta|<\beta_{cr}$. Owing to their darkness, these points could be mistaken for vortices in the laboratory if interferometric measurements are not recorded to determine the phase.

Beyond the critical phase value we find an oppositely charged pair of vortices, with the positively charged vortex remaining in close proximity to the $m=1$ vortex of Beam-1, and the negatively charged vortex diverging from the beams as the magnitude of the phase increases. Regardless of the value of $\beta$, we find the net topological charge is $m_{net}=0$, and not the sum $m_{sum}=1$. 
Fig. 4.7(B) depicts vortex trajectories as the phase is varied for a number of different values of $\sigma$. The trajectories for $\sigma_1$ and $\sigma_2$ resemble strophoids, separating regions having open and closed paths. For example, if $\sigma_1 < \sigma < \sigma_2$, a vortex dipole is created or annihilated at a point at some critical phase difference; thus the trajectory of each vortex is an open path. On the other hand, for $\sigma < \sigma_1$ or $\sigma > \sigma_2$ the dipole exists for all values of $\beta$, and the paths are closed. The right-most vortex in all these cases has a charge $m=1$.

Inspections of phase profiles indicates that the net topological charge is $m_{\text{net}}=0$ for all values of $\sigma$ and $\beta$ shown in Fig. 4.7. However, $m_{\text{net}}=1$ for the degenerate case [1], $\sigma=0$, (see left hand column in Fig. 4.6). The case $\sigma=0.13$, shown in Fig. 4.7(B), suggests that when the relative separation is small ($\sigma<<1$), one vortex having the same charge as Beam-1, circles the origin, while a second vortex of the opposite charge appears in the region beyond the effective perimeter of the beam. As a practical matter one may ignore the existence of the second vortex if it is surrounded by darkness, and in this case one may state that the charge is conserved in a weak sense. However, the second vortex is not always shrouded in darkness. For example, in Fig. 4.6(right hand column) and Fig. 4.7(B) we see that when $\sigma=1$, the two vortices may be found quite close to each other. Thus the superposition of a vortex and a Gaussian beam does not generally conserve the net value of the topological charge.
4.4 Vortex Velocity

The superposition of a vortex and Gaussian beam provides a particularly convenient means of displacing a vortex, owing to the common availability of Gaussian beams from lasers. By rapidly changing the amplitude of the Gaussian beam, the vortex may be displaced at a rate that exceed the speed of light without moving the beams or varying the phase. The speed of the vortex may be determined by applying the chain rule

\[ v_x + iv_y = \left( \frac{\partial x}{\partial A_2} + i \frac{\partial y}{\partial A_2} \right) \frac{dA_2}{dt} \]  

(4.28)

where \((x+iy)\) is given by the transcendental function, Eq. (4.21). The solution for the velocity simplifies when \(\beta = \pi/2\), in which case \(x=s, v_x=0,\) and \(v_y=(s/\sigma A_1)\exp(-4\sigma^2)\frac{dA_2}{dt}\).

The vortex traverses the beam at a speed that exceeds the speed of light when \(\frac{dA_2}{dt} > (A_2/\tau_0)\exp(4\sigma^2)\), where \(\tau_0=w_0/c\) is the time it takes light to travel the distance \(w_0\). A relatively slow light pulse may be used to achieve this if \(w_0\) is large and \(\sigma=0\) (note that any value of \(\beta\) may be used when \(\sigma=0\)). Assuming the amplitude of the Gaussian beam increases linearly with a characteristic time \(\tau_2\): \(A_2(t) = (A_2/\tau_2)t\), light speed may be achieved when \(\frac{\hat{A}_2}{A_1} > \frac{\tau_2}{\tau_0}\). This may be satisfied with a beam having a 100ps rise time and a 30mm radial size, assuming \(\frac{\hat{A}_2}{A_1}=1\).

4.5 Conclusion

Our investigation of the superposition of two coherent beams, with at least one containing an optical vortex, reveals that the transverse position of the resulting composite vortex can be controlled by varying a control parameter such as the relative phase, amplitude, or
distance between the composing beams. The three most fundamental combinations of beams were explored, namely, beams have identical charges, beams have opposite charges, and the combination of a vortex and Gaussian beam. Composite vortices were found to rotate around each other, merge, annihilate, or move to infinity. We found that the number of composite vortices and their net charge did not always correspond to the respective values for the composing beams. As may be expected from the principle of conservation of topological charge, the net composite charge remained constant as we varied the control parameters. Critical conditions for the creation or annihilation of composite vortices were determined. Composite vortex trajectories were found for special cases. The speed of motion along a trajectory was demonstrated to depend on the rate of change of the control parameter. For example we described how the speed of a vortex may exceed the speed of light by rapidly varying the amplitude of one of the beams.
Fig. 4.1: Two beams placed symmetrically with respect to the origin with fields $E_1(r_1, \phi_1), E_2(r_2, \phi_2)$ defined by equation (1).
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Fig. 4.2: Intensity patterns of a composite beam created by interference of two (+1)-charged vortex beams, which are separated by a relative distance \( \sigma = s/w_0 = 0.0, 0.63, 1.0 \) (\( \sigma \) increases in horizontal direction from left to right), for a relative phase difference of \( \beta = 0^\circ, 45^\circ, 90^\circ, 135^\circ, 180^\circ \) (\( \beta \) increases in vertical direction from top to bottom).
Fig. 4.3: Trajectories of the composite vortices shown in Fig. 3 (two single-charged vortices) as a function of $\beta$.

Fig. 4.3(a): For the case $\sigma = s/w_0 = 0.47$ direction of motion of composite vortices along with their signs is demonstrated.
Fig. 4.3(b): Trajectories are shown for several different values of $\sigma$ (0.1, 0.55, 0.707, 0.8, 1.0),
Fig. 4.3(c): Value of $\sigma$ vs $\beta_{\alpha}$. Here $\beta_{\alpha}$ is the value of $\beta$ when creation of annihilation of optical vortices occur. Above the curve we have 3 vortices [(+1), (+1) and (-1)], below – only one.
Fig. 4.4: Same as Fig. 4.2, except composite beam is created from two oppositely charged vortices, $m_1 = -m_2 = 1$. 
Fig. 4.5: Trajectories of the composite vortices shown in Fig. 5 (two oppositely-charged vortices) as a function of $\beta$.

*Fig. 4.5(a):* For the case $\sigma = s/w_0 = 0.47$ direction of motion of composite vortices along with their signs is demonstrated
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Fig. 4.5(b): Family of trajectories for different values of $\sigma$ (0.1, 0.55, 0.707, 0.8, 1.0), for $\sigma > \sigma_{\text{cr}} = 2^{-1/2}$ the trajectories become separate closed paths.
Fig. 4.6: Same as Fig. 4.2, except now composite beam is created from vortex of charge (+1) and a Gaussian beam; $m_1 = 1$, $m_2 = 0$.
Fig. 4.7: Trajectories of the composite vortices described in Fig. 7 (vortex + gaussian beam) as a function of $\beta$.

*Fig. 4.7(a)*: Again, for the case $\sigma = s/w_0 = 0.47$ direction of motion of composite vortices along with their signs is demonstrated,
**Fig. 4.7(b):** Trajectories are shown for several different values of $\sigma$ (0.13, 0.1408, 0.15, 0.50) in the vicinity of critical separation $\sigma_1 = 0.1408$. Notice the knots on trajectories where vortices are created and annihilated.
Fig. 4.7(c): Trajectories are shown for several different values of $\sigma$ (0.50, 0.627, 0.70, 1.00) in the vicinity of critical separation $\sigma_2 = 0.6271$. 
5. SPATIAL CORRELATION VORTICES: THEORY

5.1 Introduction

Two main features of an optical vortex are an azimuthally harmonic phase and a dark core. The phase structure of a vortex is due to the well-defined phase relation between different points in a vortex beam, while dark core is usually attributed to the destructive interference. In a spatially incoherent beam different points of a beam are uncorrelated and the phase structure of an optical vortex is ill defined. Likewise, incoherent beams do not interfere and the core may be expected to fill with diffused light\textsuperscript{15, 16}. As we showed in Ch. 3 using the correlation theory\textsuperscript{129-131}, the average transverse azimuthal component of the $k$-vector of a low-coherence vortex beam approaches zero as well.

In this chapter we further study the mutual coherence function (MCF) of a vortex of arbitrary topological charge, placed on a Gaussian beam. By relating the vortex phase with spatial variations in the mutual coherence function (MCF), we find robust vortex attributes that do not vanish in partially coherent light.

In Sec. 2 we review the concept of optical vortex and then analyze the MCF in the near (Sec.3) and far (Sec. 4) field. Our numerical investigation (Sec. 5) reveals the presence of Spatial Correlation Vortices (SCV’s) in the MCF, which are analogous to the composite vortices in coherent beams. An on-axis vortex in a partially coherent beam may result in the circular dislocation line in the MCF\textsuperscript{154}. We find that this dislocation breaks into two SCV’s as the conventional vortex is moved off-axis. We demonstrate
how both the coherence and the displacement of the conventional vortex in a beam affect the SCV’s.

We then derive the general analytical form of the MCF (Sec. 6) in a spatial frequency domain ($k$-space). We find, that unlike some other types of vortices\cite{22,76,184}, SCV’s and the cross-correlation in $k$-space are invariant under propagation. The calculations are found to be in agreement with our numerical analysis.

### 5.2 On- and off-axis vortices

We first rewrite the Eq. (2.1), which describes the electric field of a single optical vortex placed on a coherent scalar beam\cite{151}:

$$E(r, \phi, z, t) = A(r', z)g(r, z)\exp(i(m\phi' + \beta + 2\pi / \lambda - \omega t))$$  \hspace{1cm} (5.1)

where the transverse bipolar coordinates ($r$, $\phi$) and ($r'$, $\phi'$) are measured with respect to the center of the beam and the vortex core, respectively (see Fig. 5.1), $\omega$ is the angular frequency, $t$ is the time, integer $m$ is the topological charge, $\beta$ is an arbitrary initial phase, $z$ is the coordinate in the direction of propagation and $\lambda$ is the wavelength. We assume a paraxial approximation. We note that the field described by Eq. (5.1) for $m \neq 0$ has a helical wavefront and its total wavevector is composed of the longitudinal component $k_z$ and the transverse component $\vec{k}$. We assume that in the plane $z=0$ the beam has a Gaussian envelope: $g(r, 0) = E_0 \exp(-r^2/w_0^2)$ and a vortex core profile $A(r', 0)$, where $E_0$ and $w_0$ are a measure of the field amplitude (assumed to be real) and characteristic
beam size, respectively. We consider a wide-core vortex, shown on Fig. 5.1, with initial core profile given by \( A(r', 0) = |r'/w_0|^m \).

The zero-field point at \( r' = 0 \) is attributed to total destructive interference. We refer to this point as the vortex core. The instantaneous integrated phase along a closed path enclosing the vortex core is \( 2\pi m \). The vectors \( \vec{r} \) and \( \vec{r}' \) in the transverse plane are related by the displacement vector \( \vec{s} : \vec{r} = \vec{r}' + \vec{s} \), where \( \vec{s} \) may change with propagation. If the core coincides with the beam center (\( \vec{s} = 0 \)), then we call the beam an “on-axis” vortex, otherwise (\( \vec{s} \neq 0 \)) the beam is said to be an “off-axis” vortex. One may notice that an on-axis wide core vortex with a Gaussian envelope is a Laguerre-Gaussian mode (see Appendix A):

\[
E_n(r, \phi) = E_0 \left| r / w_0 \right|^m \exp(-r^2 / w_0^2) \exp(im\phi) \tag{5.2}
\]

We note that Eq. (5.2) is the lowest order Laguerre-Gaussian approximation of a point vortex (see Ch.2).

### 5.3 Mutual coherence function, near field

As we explained in Ch. 2&3, a partially coherent vortex beam in the initial transverse plane may be described by the MCF:

\[
\Gamma(\vec{r}_1, \vec{r}_2) = E_1(\vec{r}_1) E_2^*(\vec{r}_2) C(\vec{r}_1, \vec{r}_2) \tag{5.3}
\]

where the subscript \( j \) on \( E_j \) emphasizes that the field is a function of \( \vec{r}_j \) and the correlation properties of a light source are governed by the Gaussian-Schell correlator\(^{141} \):

\[
C(\vec{r}_1 - \vec{r}_2) = \exp\left(-\left|\vec{r}_1 - \vec{r}_2\right|^2 / l_c^2\right) \tag{5.4}
\]
where $l_C$ is the transverse coherence length in the initial plane, and $\vec{r}_1$ and $\vec{r}_2$ are arbitrary points in the beam. The general MCF for a partially coherent vortex field in the plane $z=0$ may therefore be expressed as:

$$\Gamma_m(\vec{r}_1, \vec{r}_2, z=0) = C(\vec{r}_1 - \vec{r}_2)A(r_1')A(r_2')g(r_1)g(r_2)\exp(im[\phi_1' - \phi_2'])$$  \hspace{1cm} (5.5)

where $\vec{r}_1 = \vec{r}_1' + \vec{s}$, $\vec{r}_2 = \vec{r}_2' + \vec{s}$, and notation is illustrated by the near-field part of Fig. 5.2. Eq. (5.5) may be understood as a four-dimensional field containing two collinear vortices having topological charges $+m$ and $-m$.

By choosing appropriate coordinate system we may always have vector $\vec{s}$ lay along the $x$-axis. Then the MCF of a partially coherent wide-core vortex field, described by Eq. (5.1), in the initial plane ($z=0$) becomes:

$$\Gamma_m(\vec{r}_1, \vec{r}_2, z=0) = E_0^2[(x_1 - s + iy_1)(x_2 - s - iy_2)/w_0^2]^m$$

$$\times \exp(-[r_1^2 + r_2^2]/w_0^2)\exp(-|\vec{r}_1 - \vec{r}_2|^2/l_C^2)$$  \hspace{1cm} (5.6)

where $(x_1 - s, y_1)$ and $(x_2 - s, y_2)$ are the coordinates of vectors $\vec{r}_1'$ and $\vec{r}_2'$ respectively. The MCF of a Gaussian beam ($m=0$) becomes:

$$\Gamma_{m=0}(\vec{r}_1, \vec{r}_2, z=0) = E_0^2 \exp(-ar_1^2 - ar_2^2 + br_1r_2)$$  \hspace{1cm} (5.7)

where $a = 1/w_0^2 + 1/l_C^2$ and $b = 1/l_C^2$. MCF, described by Eq. (5.6), is convenient to express as:

$$\Gamma_m(\vec{r}_1, \vec{r}_2, z=0) = [(x_1 - s + iy_1)(x_2 - s - iy_2)/w_0^2]^m\Gamma_{m=0}(\vec{r}_1, \vec{r}_2, z=0)$$  \hspace{1cm} (5.8)

Owing to the symmetry of the beam, the four-dimensional MCF may be characterized by two two-dimensional functions, which may be derived from experimental data: the intensity $I(\vec{r}) = \Gamma(\vec{r}, \vec{r})$ and the normalized cross-correlation
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\[ \chi(\vec{r}) = \frac{\Gamma(\vec{r}, -\vec{r})}{\sqrt{|I(\vec{r})I(-\vec{r})|}} \], where \( \vec{r} \) and \(-\vec{r}\) have the coordinates \((r, \phi)\) and \((r, \phi + \pi)\), respectively. We note that the absolute value, \(|\chi(\vec{r})|\), is equal to the correlator \(C(2\vec{r})\). The dark intensity core associated with a coherent vortex is known to brighten with diffuse light under decreasing coherence conditions\(^{15, 16}\). However, less is known about the normalized cross-correlation function, \(\chi(\vec{r})\).

For an on-axis vortex the phase factor in Eq. (5.5) may be expressed as \(\exp(im[\phi' - \phi_2]) = \exp(im[\phi - \phi_2]) = (-1)^m\), and thus \(\chi(\vec{r}) = (-1)^m C(2\vec{r})\) uniformly changes from a correlation to an anti-correlation state when the value of \(m\) is changed from even to odd. In particular, the value of the cross-correlation function at the origin is \(\chi(0) = (-1)^m\). This implies that the phases of the beam at \(\vec{r}\) and \(-\vec{r}\) are always in (out of) phase when \(m\) is even (odd).

For an off-axis vortex \(\chi(\vec{r}) = C(2\vec{r}) \exp(im[\phi' - \phi_2])\), and its phase structure is defined by two oppositely charged vortices located at points \(\vec{r} = \pm \vec{s}\), having topological charges \(M = \pm m\). In the limit \(\vec{s} \to 0\) these two singularities annihilate. The amplitude of \(\chi(\vec{r})\) is given by the correlator \(C(2\vec{r})\) at all points except the singularities at \(\pm \vec{s}\), where \(\chi(\vec{r})\) abruptly vanish. Near field cross-correlation vortices therefore belong to the class of “point vortices”\(^{68}\).

5.4 Far-field analysis

Both the optical field and the MCF change as the beam propagates through space. This change is most easily understood in the far-field regime \((z >> kw_0^2)\). Here we are interested in exploring how singularities affect the far-field MCF (see Fig. 5.2). The far-
field distributions of the electric field for a point vortex and a Laguerre-Gaussian vortex are easily determined. For the case \( m=1 \) the \( k \)-space representation of a point vortex beam may be described by the function\(^{68} \):

\[
E_{m=1}(k_\perp, \psi) = E_0^\infty [\left( \frac{\xi}{1.06} \right)^{2.60} + \left( \frac{\xi}{1.53} \right)^{0.86}] e^{i\psi}
\]  (5.9)

where \( \xi = k_\perp / w_0 \), \( (k_\perp, \psi) \) are transverse polar coordinates in the far-field, \( w_0 \) is the spot size of a regular Gaussian beam, \( E_0^\infty \) is the field amplitude in the far field. In our approximation we use Fourier transform of a Laguerre-Gaussian mode (5.2), which is still a Laguerre-Gaussian mode (see Appendix A):

\[
E_{m=1}(k_\perp, \psi) = A_k \xi \exp(-\xi^2) e^{i\psi}
\]  (5.10)

Both Eq. (5.9) and (5.10) have similar distributions near the vortex core. We notice the slowly decaying tail (\( 1/\xi \)) in a point vortex case in contrast with the exponential decay for a large-core vortex. For computational convenience we assume a Laguerre-Gaussian type vortex mode, and a fundamental topological charge, \( m=1 \).

Unlike the Gaussian-Shell correlator in the near field (see Eq. 5.4), we do not know \textit{a priori} the far-field correlator. Therefore the MCF in \( k \)-space must be calculated by using Fourier transform of the spatial domain distribution\(^{17} \):

\[
\Gamma_m^\infty(\vec{k}_1, \vec{k}_2, z) = \iint \Gamma_m(\vec{r}_1, \vec{r}_2, z) \exp[-i(\vec{k}_1 \cdot \vec{r}_1 - \vec{k}_2 \cdot \vec{r}_2)] d\vec{r}_1 d\vec{r}_2
\]  (5.11)

where \( \vec{k}_1 \) and \( \vec{k}_2 \) are the wavevectors in the transverse plane. Analogous to the near-field \( (\vec{r} \)-space) analysis the \( k \)-space intensity is given by \( I^\infty(\vec{k}) = \Gamma(\vec{k}, \vec{k}) \) and the normalized cross-correlation is given by \( \chi^\infty(\vec{k}) = \Gamma^\infty(\vec{k}, -\vec{k}) / [I^\infty(\vec{k})I^\infty(-\vec{k})]^{1/2} \). Numerical and analytical integration of Eq. (5.11) allow us to compute the amplitude and the phase of
the \( \chi^\omega (\tilde{k}) \). In the next section we analyze numerically calculated \( k \)-space distributions of MCF for \( m=0 \) and \( m=1 \).

### 5.5 Numerical calculations of the far-field MCF

**On-axis case**

When \( s=0 \) and \( m=0 \), the 4-dimensional integral in Eq. (5.11) may be expressed as:

\[
\Gamma_{m=0}^\infty (\tilde{k}_1, \tilde{k}_2) = E_0^2 \left( \iint T_x dx_1 dx_2 \iint T_y dy_1 dy_2 \right) \tag{5.12}
\]

where

\[
T_x = \exp(-a(x_1^2 + x_2^2) + 2b x_1 x_2) \cdot \exp[i k_{2x} x_2 - i k_{1x} x_1]
\]

\[
T_y = \exp(-a(y_1^2 + y_2^2) + 2b y_1 y_2) \cdot \exp[i k_{2y} y_2 - i k_{1y} y_1] \tag{5.13}
\]

Here \((x_1, y_1), (k_{ix}, k_{iy})\) are the coordinates of \( \tilde{r}_i \) and \( \tilde{k}_i \), and constants \( a = 1/w_0^2 + 1/l_c^2 \) and \( b = 1/l_c^2 \) are defined earlier. When \( s=0 \) and \( m=+1 \) (Laguerre-Gaussian mode), the 4-dimensional integral in Eq. (5.11) may be reduced to a linear combination of products of two-dimensional integrals:

\[
\Gamma_{m=1}^\infty (\tilde{k}_1, \tilde{k}_2) = (E_0^2 / w_0^2) \left( \iint x_1 x_2 T_x dx_1 dx_2 \iint T_y dy_1 dy_2 + \iint x_1 T_x dx_1 dx_2 \iint y_1 T_y dy_1 dy_2 + \iint y_1 T_y dx_1 dx_2 \iint y_1 T_y dx_1 dx_2 \right) \tag{5.14}
\]

The two-dimensional integrals in Eq. (5.12-5.14) were calculated numerically using a fast-Fourier transform algorithm.\(^{185}\)

Fig. 5.3 shows calculated distributions of \( I^\omega (\tilde{k}) \) and both the modulus and phase of \( \chi^\omega (\tilde{k}) \) for different values of the relative coherence length \( \sigma_c = l_c / w_0 \); \( \sigma_c = 1, 2, 5 \) and \( s = 0 \). As expected, the intensity profiles in Fig. 5.3(a, b, c) depict greater diffusion of
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light into the vortex core as the value of $\sigma_c$ decreases. On the other hand, the characteristic ring of zero correlation in the Fig. 5.3(d, e, f) suggests that this ring is a more robust feature than the dark vortex core in the intensity profile\textsuperscript{186}. This ring is attributed to a phase dislocation, as seen in Fig. 5.3(g, h, i). There the sharp transitions from black to gray indicate a $\pi$-phase shift and hence a phase dislocation. This may be interpreted as the real part of $\chi^\sigma(\tilde{k})$ changing its sign from positive (correlation) inside the dislocation ring to negative (anti-correlation) outside the ring.

Off-axis case

For $\tilde{s}=s\hat{x}$ Eq. (5.11) may again be written as a linear combination of products of two-dimensional integrals:

$$\Gamma_{n=1}^{\sigma}(\tilde{k}_1, \tilde{k}_2) = E_0^2(1/\omega_0^2) \left( \iint (x_1x_2 - s(x_2 + x_1))T_xdx_1dx_2 \iint T_ydy_1dy_2 ight. \\
+ s^2 \iint T_xdx_1dx_2 \iint T_ydy_1dy_2 + \iint T_xdx_1dx_2 \iint (y_1y_2 + is(y_2 - y_1))T_ydy_1dy_2 \\
\left. + \iint x_1T_xdx_1dx_2 \iint iy_1T_ydy_1dy_2 - \iint x_1T_xdx_1dx_2 \iint iy_2T_ydy_1dy_2 \right)$$

(5.15)

Fig. 5.4 shows the distribution of $I^\sigma(\tilde{k})$ and $\chi^\sigma(\tilde{k})$, calculated using Eq. (5.13) and Eq. (5.15) for $\sigma_c = 2$ and $s = 0.2 \omega_0$, $0.4 \omega_0$ (the $s = 0$ case is shown in Fig. 5.3(b, e, h)). As the near-field displacement in the $\hat{x}$ direction increases from $s = 0$, we see a displacement of the diffuse far-field vortex core in the $-k_y$ direction. This relation between the near and far field displacements is consistent with the results found for a coherent beam\textsuperscript{187}, and is attributed to the Gouy phase shift. At the same time the ring dislocation for the case $s = 0$ is found to break into two correlation vortices when $s \neq 0$ (see Fig. 5.4(c, d)). These have
opposite topological charges: \( M^\tau = \pm 1 \) and they appear along the \( k_y \) axis (see Fig. 5.4(e, f)).

Unlike the abruptly vanishing point vortices in \( \chi(\vec{r}) \), Fig. 5.4(c, d) depicts wide-core vortices in the far-field regime. This redistribution of \( \chi(\vec{r}) \) values indicates that the spatial coherence properties of a beam may be manipulated using an optical vortex. In particular, we may create uncorrelated areas in an otherwise coherent beam by inserting a vortex into the initial beam. These zero’s may form paths in the 4-D space of \( \vec{k}_1 \) and \( \vec{k}_2 \) (not shown).

We note that vortices in the cross-correlation may be controlled by varying the coherence length. Fig. 5.5 shows numerical results for \( s = 0.2w_0 \) and coherence length \( \sigma_c = 1, 4 \) (the \( \sigma_c = 2 \) case is shown in Fig. 5.4). For \( \sigma_c = 1 \) Fig. 5.5(c) demonstrates a virtually undistorted dislocation ring. However as we increase the coherence length (see Fig. 5.5(d)), the ring transforms into an SCV dipole. Phase profiles (see Fig. 5.5(e, f)) indicate that the dislocation ring gradually dissolves into a phase gradient forming around two singularities. However, unlike the variation of \( s \) (Fig. 5.4), increase in the coherence length moves SCV’s closer together. To explain this we note that along the former dislocation ring the cross-correlation become purely imaginary. Inside is an area of positive correlation \( \text{Re} \chi^\omega(\vec{k}) > 0 \) and outside the correlation is negative (cross-correlation phase = \( \pi \), \( \text{Re} \chi^\omega(\vec{k}) < 0 \)). If we recall that the vortex presence is associated with the anti-correlated (negative) MCF (Eq. 5.5), and vortices are phase objects that require a degree of correlation, then we may associate the incoherence with the central
area of positive correlation. Therefore, for a given $s$, the size of that area shrinks when the $\sigma_c (l_c)$ increases and vortices move closer together.

Fig. 5.6 shows the distance $k_v$ between the far-field correlation vortices as a function of the near-field vortex displacement $s$ for different values of $\sigma_c$. In the case $s = 0$, $k_v$ represents the diameter of the dislocation ring, which is inversely related to $\sigma_c$. For a coherent beam ($\sigma_c >> 1$) the ring size is zero and $k_v$ is a linear function of $s$, as explained in Ch. 4 (also see Ref. 19). As $\sigma_c$ decreases, the slope of the curve $k_v$-vs-$s$ decreases, and the anti-correlated part of the MCF moves out of the field of view, resulting in growing ring size for $s = 0$. For an off-axis vortex the decreased correlation between two composing beams results in a decreased rate of motion of SCV’s, similar to the dependence of the composite vortex velocity on the beam amplitude in the coherent case 19.

5.6 Analytical solution, Propagation

In order to calculate the MCF in an arbitrary transverse plane $z$ in a paraxial approximation one may consider MCF in $k$-space and multiply it by the free space transfer function $T(k_{1z}, k_{2z}) = \exp(i[k_{1z} - k_{2z}]z)$ 17, where $k_{1z}$, $k_{2z}$ are the $z$-components of the corresponding 3-dimensional wavevectors $\vec{K}_1$ and $\vec{K}_2$ (notice the difference between 3-dimensional wavevectors and 2-dimensional transverse plane components $\vec{k}_1$ and $\vec{k}_2$). If we assume propagation into $z>0$ plane, then $k_{1z} = (K_r^2 - k_{1x}^2 - k_{1y}^2)^{1/2}$ and
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\( \kappa_{2z} = (K^2 - k_{2x}^2 - k_{2y}^2)^{1/2} \), where \( K \) is the wavenumber, \( K = 2\pi / \lambda \), same for both wavevectors.

Earlier Carter and Wolf obtained expression for the \( k \)-space MCF for \( m=0 \) by analytical integration of Eq. (5.11)\textsuperscript{131}:

\[
\Gamma_{m=0}^\infty(\vec{k}_1, \vec{k}_2, z = 0) = E_0^2[\pi^2 / (a^2 - b^2)] \exp(-\alpha \vec{k}_1 \cdot \vec{k}_2 + 2\beta \vec{k}_1 \cdot \vec{k}_2) \]  \( (5.16) \)

where \( \alpha = a /[4(a^2 - b^2)] \) and \( \beta = b /[4(a^2 - b^2)] \). To describe the more general case \( (m \neq 0) \) we make use of the relation:

\[
i_{p+u-q-v} \frac{\partial^{p+q+u+v} \Gamma_{m}^\infty(\vec{k}_1, \vec{k}_2, z)}{\partial k_1^p \partial k_2^q \partial k_1^u \partial k_2^v} = \\
\iint [x_1^p x_2^q y_1^u y_2^v \Gamma_{m}(\vec{r}_1, \vec{r}_2, z)] \exp[-i(\vec{k}_1 \cdot \vec{r}_1 - \vec{k}_2 \cdot \vec{r}_2)] d\vec{r}_1 d\vec{r}_2
\]  \( (5.17) \)

where \( p, q, u, v \) are positive integers. Eq. (5.17) is obtained by taking partial derivatives of the left and right parts of Eq. (5.11). Eq. (5.17) may govern the calculations of the \( k \)-space form for any MCF, which can be expressed as a product of a polynomial and an MCF of a Gaussian field with a Gaussian-Shell correlator. In the case of a Laguerre-Gaussian vortex of charge \( m=1 \) we obtain:

\[
\Gamma_{m=1}^\infty(\vec{k}_1, \vec{k}_2, z) = \frac{1}{w_0^2} \left[ \frac{\partial^2}{\partial k_{1x} \partial k_{2x}} + \frac{\partial^2}{\partial k_{1y} \partial k_{2y}} + s^2 - is \left( -\frac{\partial}{\partial k_{1x}} - \frac{\partial}{\partial k_{2x}} \right) + \\
s \left( \frac{\partial}{\partial k_{1y}} + \frac{\partial}{\partial k_{2y}} \right) + i \left( \frac{\partial}{\partial k_{2x}} \frac{\partial}{\partial k_{1y}} - \frac{\partial}{\partial k_{1x}} \frac{\partial}{\partial k_{2y}} \right) \right] \Gamma_{m=0}^\infty(\vec{k}_1, \vec{k}_2, z) \]  \( (5.18) \)

Calculation of the derivatives and multiplication by the transfer function \( T(k_{1z}, k_{2z}) \) result in:
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\[
\Gamma_{m=1}^{x}(\vec{k}_1, \vec{k}_2, z) = \frac{1}{w_0^2} \{4[\beta + (\alpha^2 + \beta^2)\vec{k}_1 \cdot \vec{k}_2 - \alpha\beta(k_1^2 + k_2^2) + i(\alpha^2 - \beta^2)(k_{2x}k_{1y} - k_{1x}k_{2y})] + 2s[(\beta - \alpha)(k_{1y} + k_{2y}) - i(\alpha + \beta)(k_{2x} - k_{1x})] + s^2 \} 
\times \exp[i((K^2 - k_{1x}^2 - k_{1y}^2)^{1/2} - (K^2 - k_{2x}^2 - k_{2y}^2)^{1/2})z] \Gamma^{x}_{m=0}(\vec{k}_1, \vec{k}_2, 0) \tag{5.19}
\]

where we assume propagation into \(z>0\) plane.

Substitution of \(\vec{k}_1 = \vec{k}\) and \(\vec{k}_2 = -\vec{k}\) \((k_{1z} = k_{2z})\) gives us the cross-correlation function\(^{151}\):

\[
\Gamma_{m=1}^{x}(\vec{k}, -\vec{k}, z) = \frac{1}{w_0^2} \{4[\beta + (\alpha^2 + \beta^2)k_z^2 + 4is(\alpha + \beta)k_z + s^2] \Gamma^{x}_{m=0}(\vec{k}, -\vec{k}, 0) \tag{5.20}
\]

where \((k_x, k_y)\) are \(\vec{k}\) coordinates. We now substitute the beam size \(w_0\) and the transverse coherence length \(l_C\) into coefficients \(\alpha\) and \(\beta\):

\[
\Gamma_{m=1}^{x}(\vec{k}, -\vec{k}, z) = \frac{(w_0^2 / 4)}{(-k^2 + 4isk_x / w_0^2 + k_y^2)} \Gamma^{x}_{m=0}(\vec{k}, -\vec{k}, 0) \tag{5.21}
\]

where

\[
k_y = 2(1 / \sigma + s^2 / w_0^4)^{1/2} \tag{5.22}
\]

and \(\sigma = l_C^2 + 2w_0^2\) is the beam spread. We note the right part of Eq. (5.21) is independent of \(z\) and therefore the cross-correlation function in the \(k\)-space is invariant upon propagation. Further analysis of Eq. (5.21) for \(s = 0\) confirms the presence of a characteristic ring\(^{154}\) of radius \(k_r = 2 / \sigma^{1/2}\). In the case \(s \neq 0\) the cross-correlation contains two zeros\(^{151,188}\), located at points \((0, \pm k_y)\).

One may verify that the cross-correlation is singular at these points by rewriting polynomial part of Eq. (5.21) in terms of new coordinates \((k'_x, k'_y) = (k_x, k_y \pm k_y)\):

\[
\Gamma_{m=1}^{x}(\vec{k}, -\vec{k}, z) = \{-(w_0^2 / 4)(k_x^2 + k_y^2) + isk'_x \pm k'_yk_y\} \Gamma^{x}_{m=0}(\vec{k}, -\vec{k}, 0) \tag{5.23}
\]
In the vicinity of zeros the polynomial term becomes \( isk'_x \pm k'_yk_y \), defining vortices of charge \( m = +1 \) at \((k_x, k_y) = (0, k_y)\) and \( m = -1 \) at \((k_x, k_y) = (0, -k_y)\).

Fig. 5.7 shows the unit-less distance \( d = 2k_yw_0 \) between zeros in the cross-correlation as a function of \( s/w_0 \) and \( l_c/w_0 \), where \( k_y \) is given by Eq. (5.22) (for \( s=0 \) the unit-less diameter of the characteristic ring \( 2k_yw_0 \) is shown instead). Analysis of Eq. (5.22) reveals that in the coherent limit \( (l_c \to \infty) \sigma \to 0 \) and \( k_y \) is a linear function of vortex displacement \( s \):

\[
k_y = 2s/w_0^2
\]

(5.24)

In the incoherent limit \( (l_c \to 0) \) the beam spread is defined by the beam size, \( \sigma \to 2w_0^2 \), and we obtain:

\[
k_y = (4s^2 + 2w_0^2)^{1/2}/w_0^2
\]

(5.25)

In the limit \( s \ll w_0 \) the first-order expansion of Eq. (5.25) indicates that \( k_y \) is a quadratic function of \( s \): \( k_y \approx (2^{1/2}/w_0)(1+s^2/w_0^2) \). We note that for a beam of finite size \( k_y \) always remain finite. This result means that from the mathematical standpoint the phase singularities in a cross-correlation function may exist even in the incoherent limit \( (l_c \approx \lambda) \).

We note that in order to obtain \( x \)-space MCF in an arbitrary plane \( z \) one may calculate \( k \)-space MCF distribution in that plane and calculate an inverse transform of \( \Gamma_m^{x}(\vec{k}_1, \vec{k}_2, z) \). The general solution has the form:
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\[
\Gamma_m^\infty (\vec{r}, \vec{r}, z) = \iiint \Gamma_m^\infty (\vec{k}_1, \vec{k}_2, z = 0) \\
\exp(i[(K^2 - k_{1x}^2 - k_{1y}^2)^{1/2} - (K^2 - k_{2x}^2 - k_{2y}^2)^{1/2}]z) \\
\exp(i[k_{1x} x_1 + k_{1y} y_1 - k_{2x} x_2 - k_{2y} y_2]dk_{1x} dk_{1y} dk_{2x} dk_{2y}) 
\]

(5.26)

where for a Gaussian beam (\(m=0\)) and a vortex (\(m=1\)) \(\Gamma_m^\infty (\vec{k}_1, \vec{k}_2, z = 0)\) is given by Eq. (5.16) and (5.19) respectively.

5.7 Conclusion

The main results obtained in this chapter are the derivation of the general form of the mutual coherence function of a wide-core vortex beam in an arbitrary transverse plane and analysis of the MCF in near and far fields. Spatial correlation vortices in the cross-correlation function were found and investigated. Motion and transformation of SCV’s were found to depend on the coherence length and on the off-axis displacement of the original vortex. We found point and large-core vortices in the near field and in the far-field cross-correlation distributions respectively. We show that singularities in the cross-correlation may exist even in the incoherent limit. We also demonstrate that the cross-correlation function shape is invariant upon propagation, indicating that pairs of spatial correlation vortices are stable features of a beam and may persist even when coherence properties of a beam change with propagation.

The redistribution of the cross-correlation magnitude due to the vortex presence may allow selective control over the spatial coherence properties of the beam. Possible applications include optical coherence tomography, where control of the correlation may allow to suppress coherent scattering from undesirable areas. Our ability to create,
annihilate and move a vortex by varying the coherence length may be employed in optical filtering and optical switching.

Fig. 5.1: Off-axis optical vortex. Intensity profile and the phase map. Coordinates \((r, \phi)\) are with respect to the beam center. Coordinates \((r', \phi')\) are with respect to the vortex core center.
Notation of the correlation function of a pair of points. (a) In $r$-space (near field) the Gaussian beam with a center at point $(0, 0)$ has an embedded vortex with the core at point $(s, 0)$. Coordinates $\vec{r}_1$ and $\vec{r}_2$ are with respect to the origin and $\vec{r}_1'$ and $\vec{r}_2'$ are with respect to the vortex core center. (b) The $k$-space (far-field) distribution is given by the Fourier transform of the near-field. MCF in the $k$-space is a function of coordinates $\vec{k}_1$ and $\vec{k}_2$.

When original beam contains an off-axis vortex on the $x$-axis, far-field distribution of the cross-correlation function may contain oppositely charged vortices at points $(0, k_V), (0, -k_V)$. 
**Fig. 5.3:** On-axis vortex case; relative coherence length $\sigma_c = 1, 2, 5$. Numerically calculated intensity profiles (a, b, c); modulus (d, e, f) and phase (g, h, i) of the normalized cross-correlation in k-space. Phase map distribution uses black color to show zero phase value and white color to show $2\pi$ phase value.
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\[ \sigma_c = 2 \]

\[ k_y \]

\[ k_x \]

\[ s = 0.2w_0 \]

\[ s = 0.2w_0 \]

\[ s = 0.4w_0 \]

\[ s = 0.4w_0 \]

**Fig. 5.4:** Off-axis vortex as a function of the off-axis displacement \( s \); \( \sigma_c = 2 \); \( s = 0.2w_0 \), \( 0.4w_0 \). Numerically calculated intensity profiles (a, b, c); modulus (d, e, f) and phase (g, h, i) of the normalized cross-correlation in k-space.
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Fig. 5.5: Off-axis vortex as a function of coherence length; $s = 0.2w_0$; $\sigma_c = 1, 4$.

Numerically calculated intensity profiles (a, b, c); modulus (d, e, f) and phase (g, h, i) of the normalized cross-correlation in k-space.
Fig. 5.6: Far-field separation $k_v$ between spatial correlation vortices in the normalized cross-correlation for different values of $s$ (the near-field vortex displacement) and $\sigma_c = l_c / w_0$ (the relative coherence length); $w_0$ and $w'_0$ are the sizes of a Gaussian envelope in the near and in the far field respectively. Marks show numerically calculated points, curves are plotted using interpolation.
Fig. 5.7: Relative far-field separation $k_r w_0$ between spatial correlation vortices in the cross-correlation as a function of relative displacement $s/w_0$ (the near-field vortex displacement) and the relative coherence length $l_c/w_0$; $w_0$ is the size of a Gaussian envelope in the near field.
6. SPATIAL CORRELATION VORTICES: EXPERIMENT

6.1 Introduction
In this chapter we report the results of experimental investigation of the cross-correlation function using a wave-front folding interferometer. Spatial Correlation Vortices (SCV’s) in the cross-correlation function are verified to exist in the low-coherence regime, when the dark core of a traditional vortex beam fills with diffuse light. Locations of SCV’s were verified by moving a vortex in the original beam. Sec. 2 is a brief clarification on how to place an optical vortex in a beam. Sec. 3 explains how to find a cross-correlation distribution with a wave-front folding interferometer. Experimental setup description and results are presented in Sec. 4. We also present a simple geometrical model of our observations in Sec. 5, followed by conclusion.

6.2 Optical vortex and a phase mask
Experimentally a vortex in an optical beam may be created by transmitting a planar [Gaussian] beam \((m = 0)\) through the phase mask of variable thickness
\[
d = d_0 + m\lambda_0\phi/2\pi(n_s-n_0),
\]
where \(d_0\) is the minimum thickness, \(\lambda_0\) is the intended wavelength and \(n_s\) and \(n_0\) are indices of refraction of the substrate and of the surrounding medium. The amplitude distribution at the mask corresponds to that of a point vortex and therefore, as we mentioned in Ch. 5, our use of a wide-core vortex in numerical
simulation is an approximation. We note however that a point vortex may be transformed into a wide-core vortex using spatial filtering and that in a far field both vortices have similar amplitude distributions near the core.

6.3 Wavefront-Folding Interferometer

As we explained in Ch.5, the four-dimensional MCF may be characterized by two two-dimensional functions: the intensity \( I(\vec{r}) = \Gamma(\vec{r}, \vec{r}) \) and the cross-correlation \( X(\vec{r}) = \Gamma(\vec{r}, -\vec{r}) \). Experimentally the normalized cross-correlation \( \chi(\vec{r}) = X(\vec{r})/[I(\vec{r})I(-\vec{r})]^{1/2} \) may be analyzed using the wavefront-folding interferometer (WFFI) (see \(^{189}\) and \(^{17}\), p.163). For a stationary quasi-monochromatic field \( E(\vec{r}) = E(x, y) \) in a plane \( z \) this type of interferometer allows to interfere fields \( E(-x + x_c, y + y_c) \) and \( E(x + x_c, -y + y_c) \), where \( (x_c, y_c) \) are the Cartesian coordinates of the interferometer center of symmetry. In our investigation we limit ourselves to this center of symmetry being at the origin. The interferogram intensity then has the form:

\[
I_{\text{int}}(x, y) = I(-x, y) + I(x, -y) + 2 \text{Re}\{<E(-x, y)E^*(x, -y)\exp(ik\Delta)\}>\cos(k[c_x + c_y, y])
\]

(6.1)

where \( \Delta \) is the length difference between interferometer arms, and \( c_x, c_y \) are constants defined by the interferometer alignment. For a slowly varying field distribution we may define \( I_{\text{max}}(x, y) \) and \( I_{\text{min}}(x, y) \) as local maximum and minimum intensities of the fringe pattern. The fringe visibility, defined as

\[
\eta(x, y) = (I_{\text{max}}(x, y) - I_{\text{min}}(x, y))/(I_{\text{max}}(x, y) + I_{\text{min}}(x, y))
\]

is related to the cross-correlation \( X(x, y) \) as:
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\[ \eta(x, y) = 2 \text{Re}\{X(x, -y) \exp(i k \Delta)\}/\{|I(-x, y) + I(x, -y)\} \]  

(6.2)

Assuming \( I(-x, y) \approx I(x, -y) \), real and imaginary parts of the normalized cross-correlation \( \chi(\vec{r}) \) are given by the visibility \( \eta(x, y) \) for \( \Delta = 0, \lambda/4 \) respectively. The phase features of \( \chi(\vec{r}) \) may be determined from the fringe deformation.

6.4 Experimental Observation of Spatial Correlation Vortices

As we explained earlier, a vortex may be created by transmitting a beam with a planar phase front through the phase mask of variable thickness. A schematic diagram of partially coherent light transmitted through a vortex phase mask is shown in Fig. 6.1. A vortex mask of diameter \( 2w \) is placed in the plane \( z = 0 \) at a distance \( z_s \) from an extended incoherent source of diameter \( D_s \). The transverse coherence length at the mask is approximately given by \( l_c = 1.28 \lambda z_s / D_s \), where \( \lambda \) is the wavelength of a source.

The MCF in the detection plane, located at a distance \( z_d >> \lambda \) from the phase mask (see Fig. 6.2), for a narrow-band source and a small-angle approximation\(^{18} \) is approximately given by\(^{18} \):

\[ \Gamma(\vec{r}_{d1}, \vec{r}_{d2}, z_d) \approx (z_d / \lambda)^2 \iint_{z=0} \Gamma(\vec{r}_1, \vec{r}_2, 0) \exp[i k (R_1 - R_2)] R_1^{-2} R_2^{-2} d\vec{r}_1 d\vec{r}_2 \]  

(6.3)

where \( \Gamma(\vec{r}_1, \vec{r}_2, 0) \) may be described by Eq. (5.5), \( \vec{r}_{d1} \) and \( \vec{r}_{d2} \) are two points in the detection plane, \( R_i^2 = |\vec{r}_i - \vec{r}_{d1}|^2 + z_d^2 \) \((i=1,2)\), and \( k \) is the wave number, \( k = 2\pi / \lambda \).

Our experimental setup is shown in Fig. 6.3. Broadband light from an incoherent halogen source passes through two apertures. The spatial coherence is controlled by varying the aperture \( Ap \) radial size \( A \) (0.25mm to 1.5mm with an increment of 0.125mm).
and precision of ±5nm), while the second aperture \( Ap_2 \) has a fixed radial size \( w = 2.5\text{mm} \). The distance between the apertures is \( B = 42\text{cm} \). The transverse coherence width \( l_c \) at the second aperture is approximately given by \( l_c = 0.64(\bar{\lambda}z_s / A)^{17} \), where \( \bar{\lambda} \) is the average wavelength. Aperture \( Ap_2 \) is imaged with unity magnification by the lens system \( L_1 \) of focal length \( f_1 = 25\text{mm} \) into a vortex phase mask (PM) plane \( z=0 \), designed to produce an \( m = 1 \) vortex at wavelength \( \lambda = 890\text{nm} \) and air interface. Phase mask transverse position in the plane is controlled with a micrometer, allowing on- and off-axis center location with a precision of 1nm. The resulting beam is passed through the WFFI, which is built from a standard Mach-Zehnder interferometer by incorporating two Dove prisms \( DP_1 \) and \( DP_2 \) to fold the wavefront in each arm, one about the horizontal, and second about the vertical axis. The interferometer is aligned for zero path difference between the arms to allow for the white light interference and investigation of \( \text{Re} \chi(x_d, -y_d) \) in the detection plane \( z = 92\text{mm} \). The detection plane is imaged with a magnification of 3 by two lenses having focal lengths, \( f_2 = 125\text{mm} (L_2) \) and \( f_3 = 500\text{mm} (L_3) \) onto a cooled CCD camera (Mead model 416XT). A 50nm band pass filter with a mean transmitted wavelength of \( \bar{\lambda} = 800\text{nm} \) is attached to the front of the camera to achieve quasi-monochromaticity and minimize temporal coherence effects.

Observed interference fringes allow for the forking analysis\(^4\) of the cross-correlation. The fringe visibility is appreciably different from zero inside the coherence area\(^{17} \), which border (ring of zero fringe visibility) may be defined by the Airy disk\(^{190} \). The coherence area location and diameter are verified by removing the phase mask from a beam. We analyze the fringe structure inside the Airy disk as a function of aperture \( Ap_1 \).
size $A$ and off-axis displacement $s$ of the vortex phase mask. Fig. 6.4 illustrates the experimentally obtained distribution of intensity, defined by Eq. (6.1) for $\Delta = 0$, for the partially coherent light source with a charge +1 embedded vortex, $a_1 = 0.5 \text{mm}$ (see Fig. 6.4(a),(b)) and $a_1 = 1 \text{mm}$ (see Fig. 6.4(c),(d)). For an on-axis vortex ($s = 0$, see Fig. 6.4(a),(c)) we observe a dislocation ring, not present for a plain-wave beam. We note that zero fringe visibility at the ring indicates zero of the real part of MCF. As we move the vortex off-axis (displacement $s > 0$), the dislocation ring breaks into two oppositely charged composite spatial correlation vortices (see Fig. 6.4 (b),(d)), identified by the vertex of fringe forks. We note that the distance between SCV’s is larger for the more coherent beam (see Fig. 6.4 (b)) illustrating SCV’s dependence on the coherence length of a beam.

One may notice that due to the wavelength mismatch between our band pass filter ($\lambda = 800\text{nm}$) and the phase mask ($\lambda = 890\text{nm}$) additional vortices may form inside the coherence area (see Fig. 6.4(a), quadrants 1&3). That effect is not related to the coherence properties and may be attributed to fractional vortices.$^{164}$

Fig. 6.5 shows the displacement of SCV’s from the center of symmetry as a function of $s$ (if dislocation ring is formed instead, the ring radius is shown). Experimental data (marks) are compared with numerical results, obtained in$^{151}$(curves). Numerical data were calculated in the far field for a Laguerre-Gaussian vortex$^{151}$ and converted to metric terms using experimental values. Since in Chapter 5 we were only able to calculate vortex separation for $s/w_0 < 0.6$, here we used 4-th order polynomial interpolation to calculate $d$ for larger $s/w_0$. The behavior of SCV’s in our experiments and
simulations are qualitatively similar for $s/w_0<0.5$. The growing difference at large values of $s$ may be attributed to our use of Laguerre-Gaussian approximation in numerical calculations and, more fundamentally, to the use of Gaussian-Schell correlator. While the Gaussian-Schell model is robust for low values of $s$, for large separation values the presence of an Airy disk in the experimentally obtained cross-correlation indicates that the actual coherence properties at vortex locations may differ significantly from the model. Correct analysis may require Bessel-type correlator. We also note that the experimental and numerical data are obtained in different transverse planes. However as we have shown in Ch. 5, SCV’s separation in $k$-space may be expected to remain invariant upon propagation, and thus in $x$-space it may be described by Eq. (2.4) for the beam spread. The main qualitative difference between two planes is expected to be the vortex rotation due to Gouy phase shift (for the coherent case see $^4$, p.117).

For a completely coherent beam Fig. 6.5 indicates that the vortex dislocation ring size is zero (spatial singularity annihilate), while SCV’s turn into composite vortices, described in Ch. 4. As the coherence length decreases, the dislocation ring size for an on-axis vortex increases$^{154}$. This may be understood if we recall (Ch. 5) that real part of $\chi(\vec{r})$ for a vortex beam of topological charge $m=1$ is negative$^{151}$. The vortex presence may be associated with the anti-correlated (negative real part) cross-correlation. As the coherence decreases, the area of positive correlation in the center of cross-correlation grows in size and negative part moves out of the field of view, resulting in growing ring size / vortex separation, while the rate of motion of SCV’s $[\partial d/\partial s]$ decreases.
6.5 Geometrical Model Interpretation

Fig. 6.1 provides a geometrical description of partially coherent light propagating through an off-axis vortex phase mask. The transverse plane coordinates of a center of a mask are \((s, 0)\). In the detection plane \(z = z_d\) a conical projection of the source through the center of a mask forms a circular region of radius \(r_d = D_s(z_d/z_s)\) with the center at point \((s_d, 0)\), where \(D_s\) is the size of a source, \(z_s\) is the distance between source and mask planes, and \(s_d = s(1 + z_d/z_s)\). Light collected inside this region contains rays from all sectors of the mask, while an outside observer doesn’t see the mask center and collects rays from only adjacent sectors of the mask.

Cross-correlation in the detection plane may be illustrated using Fig. 6.2. The light at points \((x_d, y_d)\) and \((-x_d, -y_d)\) may be represented by projections of the source ‘Beam1’ and ‘Beam2’, the second projection is through an “imaginary mask”, located at the point \((-s, 0)\) in plane \(z = 0\). In the coherent limit \((D_s \rightarrow \infty)\) \(r_d \rightarrow 0\) and the cross-correlation contains singularities at points \((\pm s_d, 0)\). As the coherence decreases, \(r_d\) grows and now points \((\pm s_d, 0)\) are filled with light from all segments of a corresponding phase mask. Careful calculations (see Ch. 5 and \(^{151}\)) show that SCV’s are displaced in the horizontal direction to a new location further away from each other.

6.6 Conclusion

Spatial correlation vortices were detected in the mutual coherence function using a wavefront folding interferometer and forking analysis. SCV’s behavior was analyzed and
found to depend on the coherence length. We showed that SCV’s response to the changes in off-axis displacement of original vortex is determined by the coherence length and varies as the SCV’s move outside the central area of the cross-correlation function.
Fig. 6.1: A diagram of partially coherent light transmitted through a phase mask.

Light from an incoherent light source of transverse size $D_s$ in the plane $z = z_s$ propagates to the phase mask plane $z=0$. The center of the mask is displaced from the optical axis in horizontal direction by distance $s$. In the plane $z = z_d$ a conical projection of the source through the center of the mask forms an enclosed circular region.
Cross-correlation is the correlation between points \((x_d, y_d)\) and\((-x_d, -y_d)\), and may be described by interfering two symmetrical projections ‘Beam1’ and ‘Beam2’.
An experimental setup shows light from a spatially incoherent light source (halogen bulb) with a mean wavelength of 800nm and a bandwidth of 50nm passing through two apertures separated by a distance $B = 42$cm. The $Ap_2$ plane is imaged with unity magnification by a lens system $L_1$ (focal length $f_1 = 25.1$mm) onto a phase mask PM, designed to produce an $m=1$ vortex at wavelength $\lambda = 890$nm and air interface. Phase mask transverse location is controlled with a micrometer, allowing on- and off-axis mask center location. The plane $z=92$mm is imaged with a magnification of 3 by two lenses having focal lengths, $f_2 = 125$mm ($L_2$) and $f_3 = 500$mm ($L_3$) onto a cooled CCD camera (Mead model 416XT) after passing through a wavefront folding interferometer comprising Dove prisms, $DP_1$ and $DP_2$. 

\[
\text{Fig. 6.3:}
\]
Fig. 6.4: Experimental interferograms of the cross-correlation function of a vortex beam of charge $m=1$. Partially coherent light at the phase mask has a radial size $w=2.5$mm, and relative coherence length $l_c/w$ of 0.34 (a-b) or 0.17 (d-f). The imaged plane is located at a distance of 92mm from the phase mask. The phase mask off-axis displacement $s$ is: (a, c) 0µm, on-axis vortex, dislocation ring is observed; (b, d) 7µm, off-axis vortices in the fringe pattern are present.
Distance between spatial correlation vortices $d$ vs. the off-axis displacement $s$ for coherent and partially coherent light for partially coherent and highly coherent ($l_c/w=0.17; 0.34; 340$) beams. Markers show experimental points, curves – results of numerical modeling.
7. SPONTANEOUS DOWN CONVERSION

7.1 Introduction

In previous chapters we considered a partially coherent beam with a vortex embedded into it, i.e. an already partially coherent beam was sent through a vortex phase mask. In this last chapter we consider an example of a process where a coherent vortex beam is sent through an optical system, which then affects the beam’s statistical properties.

In recent years the investigation of parametric scattering regained the interest, in particular due to the question of what happens to an optical vortex and associated orbital angular momentum of light in a 3-photon processes. Here we consider the degenerate down-conversion, where a pump photon of frequency $\omega_{\text{pump}}$ is split into signal and idler photons of the same frequency $\omega_{\text{signal}} = \omega_{\text{idler}}$. In Sec. 2 we shall briefly discuss the orbital angular momentum of photons in a vortex beam. Sec. 3 will explain the phase-matching conditions, which govern the process of down-conversion. In Sec. 4 we shall present quantum and wave models, which may be used to find spatial distributions of signal and idler beams. Numerical results are presented in Sec. 5, followed by conclusion.
7.2 Optical Vortex and Orbital Angular Momentum

The vortex phase factor \( \exp(im\phi) \), where \( m \) is the topological charge (\( m = 0, \pm 1, \pm 2, \ldots \)), and the resulting helical phase front of beam are responsible for the orbital angular momentum (OAM), carried by photons in a vortex beam. For example, each photon in a Laguerre-Gaussian (LG) mode beam, described in the transverse plane \((r, \phi)\) by:

\[
E_m(r, \phi) = E_0 \sqrt{r}/w_0 |m| \exp(-r^2/w_0^2) \exp(im\phi) \quad (7.1)
\]
carries \( m \) quanta of OAM. Here we follow notation of Ch. 2, where \( w_0 \) is defined as the size of a Gaussian beam.

In classical depiction, schematically shown on Fig. 7.1 the OAM of photons in a vortex beam may be understood by expressing the transverse \( k \)-vector component of a beam as a gradient of the phase:

\[
\vec{k}_\perp = -\nabla \phi = -\hat{\phi} m / r \quad (7.2)
\]
where \( \hat{\phi} \) is the azimuthal unit vector. We may then express the mechanical angular momentum of a photon as:

\[
\vec{L} = \hbar \vec{r} \times \vec{k}_\perp = \hbar m \hat{z} \quad (7.3)
\]
where \( \hat{z} \) is the axial unit vector (normally the direction of beam propagation).
7.3 Phase-Matching Conditions

A 3-photon process of spontaneous down-conversion is a conversion of a pump photon into signal and idler photons as a result of a non-linear interaction with a crystal (see diagram on Fig. 7.2). The so-called phase-matching conditions, recognized as the conservation laws for the momentum and energy, may be used to govern the process of conversion. We define the wave-vector mismatch and the frequency mismatch as:

\[ \Delta \tilde{k} = \tilde{k}_p - \tilde{k}_s - \tilde{k}_i \]

\[ \Delta \omega = \omega_p - \omega_s - \omega_i \]

respectively, where \( \tilde{k}_p, \tilde{k}_s, \tilde{k}_i \) are \( k \)-vectors for the pump, signal, and idler waves, and \( \omega_p, \omega_s, \omega_i \) are the corresponding frequencies.

The momentum conservation results in the wave-vector matching condition:

\[ \Delta \tilde{k} = 0 \quad (7.4) \]

and energy conservation \( \hbar \omega_p = \hbar \omega_s + \hbar \omega_i \) results in the frequency-matching condition:

\[ \Delta \omega = 0 \quad (7.5) \]

Together Eq. (7.4) and (7.5) describe the conservation of the wave phase term \( (\tilde{k} \cdot \tilde{r} - \omega t) \) in a sense that the phase of a pump wave is equal to the sum of phases of signal and idler waves.

We limit ourselves to the process of degenerate parametric scattering, i.e. \( |\omega_p| = |\omega_i| \), and start with an assumption that the OAM is also conserved. Conservation of OAM may be expressed as the topological charge conservation condition, which states that the topological charge \( m_p \) of a pump beam equals to the sum of topological charges of signal \( (m_s) \) and idler \( (m_i) \) beams. In other words, in the process of conversion the topological charge mismatch \( \Delta m = m_p - m_s - m_i \) is expected to be zero:
\[ \Delta m = 0 \] (7.6)

We may define the total phase of a beam as:

\[ \Phi = \vec{k} \cdot \vec{r} - \omega t + m\phi \] (7.7)

In the case when all three phase-matching conditions are satisfied, the total phase of a pump wave is equal to the sum of total phases of signal and idler waves:

\[ \Delta \Phi = \Phi_p - \Phi_s - \Phi_i = \Delta \vec{k} \cdot \vec{r} - \Delta \omega t + \Delta m\phi = 0 \] (7.8)

where \( \Phi_p, \Phi_s, \Phi_i \) are phases of pump, signal, and idler waves respectively.

Immediate question that arises is what happens to a vortex pump beam of topological charge \( m_p = 1 \). Given that topological charge of a vortex beam is an integer quantity, the topological charge is expected to be distributed between signal and idler waves in an asymmetric fashion. Physically the quantum of OAM from the pump photon may be expected to go randomly to the signal or idler photons.

We note two different types of phase-matching available for investigation. In both cases the pump wave has to be extraordinary polarized. In a type-I process both signal and idler waves have extraordinary polarization: \( e_p = e_s + e_i \), while in a type-II process the signal wave is extraordinary polarized and the idler wave has ordinary polarization: \( e_p = e_s + o_i \), where \( e \) and \( o \) denote an extraordinary and ordinary waves. The major advantage of the type-II phase-matching is the possibility to distinguish photons in signal and idler beams using a simple polarizer. When desired, signal and idler beam can also be made indistinguishable by using a 45° polarizer. For that reason in our experiments and simulations we use type-II phase-matching.
7.4 Signal spectrum

We assume that the amplitude of a pump field inside the crystal is described by the Laguerre-Gaussian mode profile, given by Eq. (7.1). We also assume the non-focusing regime and non-depletion by the conversion process, i.e. constant pump beam transverse profile along the axial direction $z$. That assumption is valid in a case of thin crystals and low conversion efficiency. A down-conversion signal may be described using the angular ($k$-space) distribution. In a general case the spatial-temporal spectrum of signal photons is given by\textsuperscript{121}:

$$g(\omega, \vec{k}_s) = C_1 \int \delta(\Delta \omega) \left| f(\Delta \vec{k}) \right|^2 d\vec{k}_i$$  \hspace{1cm} (7.9)

where integration is conducted via the idler $k$-space volume $V$ of a non-linear crystal, $\delta(\Delta \omega)$ is a delta-function and $f(\Delta \vec{k})$ is the conversion form-factor of the crystal. The shape of $f(\Delta \vec{k})$ is determined by the crystal properties, as well as by the pump field. In a simplest case, when microscopic properties of a crystal are defined by the constant macroscopic susceptibility $\chi$, $f(\Delta \vec{k})$ may be obtained from Fourier-transformation of the pump field inside the crystal over the $k$-vector mismatch:

$$f(\Delta \vec{k}) = \int_{-\infty}^{\infty} \exp(i \Delta \vec{k} \cdot \vec{r}) E_p(\vec{r}) d\vec{r}$$  \hspace{1cm} (7.10)

Integration in Eq. (7.9), which describes the photon spectrum, is conducted over the photon form-factor $\left| f(\Delta \vec{k}) \right|^2$, and any information about the phase information of a pump wave is removed by taking the modulus of $f(\Delta \vec{k})$. Hence the photon spectrum has the dimension of intensity. Experimentally that type of spectrum may be obtained by
photon-counting setups. In contrast, the field distribution may be obtained by directly integrating over \( f(\Delta \vec{k}) \):

\[
h(\omega_i, \vec{k}_i) = C_i \int \delta(\Delta \omega) f(\Delta \vec{k}) d\vec{k}_i
\]

We note the important difference between intensities, calculated using Eq.’s (7.9) and (7.11). The first one provides the number of photons registered by the detector, whereas the second provides the field. Therefore the intensity, calculated from Eq. (7.11) by taking squared modulus, allows taking into account interference effects. If the process of parametric scattering is completely random (such as when the molecules are moving chaotically) and down-conversion signal waves, produced by different combinations of \( k \)-vectors, do not interfere, then both models may be expected to produce similar results. However in a non-linear crystal one may expect a degree of correlation between various above-mentioned waves and thus the use of quantum theory of optical fields\(^{192} \) may be necessary. In this work we attempt to use the model, described by Eq. (7.11), to consider the parametric scattering of beams with helical phase fronts.

Radial distributions of \( f(\Delta \vec{k}) \), calculated using Eq. (7.10) for Laguerre-Gaussian modes \( m = 0 \) (Gaussian beam) and \( m = 1 \) (vortex), and normalized to \( f(0) \) of a Gaussian beam, are shown in Fig. 7.3. The distributions are plotted as a function of the transverse radial component \( \Delta \vec{k}_{\perp,\rho} \) of wave-vector mismatch. As mentioned earlier, we assume constant conversion efficiency throughout a non-linear crystal for a given pump field amplitude, \( \vec{k} \) and wavelength. We note that for \( m = 1 \) the from-factor \( f(\Delta \vec{k}) \) has a minimum at the origin: \( f(0) = 0 \). Therefore in a vortex case the conversion efficiency at
\[ \Delta \bar{k} = 0 \]  is zero and a conversion is only possible due to the ‘wings’ of \( f(\Delta \bar{k}) \). In the following calculations we assume a monochromatic detection system, which only detects waves at half the pump wave frequencies. In that case the frequency phase-matching condition is automatically satisfied.

### 7.5 Quantum and Wave Models

For convenience we refer to the photon-counting model, described by Eq. (7.9), as *quantum picture*, as opposed to *wave picture*, described by Eq. (7.11). Signal beam intensity distribution, obtained using quantum picture at \( \omega_s = \omega_p / 2 \), has the form:

\[
I_s(\bar{k}_s) = \iiint_{k_s-space} |f(\Delta \bar{k})|^2 d\bar{k}_i
\]

(7.12)

In contrast the wave picture gives us the field:

\[
\bar{E}_s(\bar{k}_s) = \iiint_{k_s-space} f(\Delta \bar{k})d\bar{k}_i
\]

(7.13)

and intensity is given by \( \bar{I}_s(\bar{k}_s) = |\bar{E}_s(\bar{k}_s)|^2 \). Both models may take advantage of the slowly-varied amplitude approximation:

\[
\Delta \bar{k} = f_{x_{xy}}(\Delta k_x, \Delta k_y) f_z(\Delta k_z)
\]

(7.14)

where transverse component of \( f(\Delta \bar{k}) \) is given by:

\[
f_{xy}(\Delta k_x, \Delta k_y) = \iint_{xy-plane} E_p(x, y) \exp i(\Delta k_x x + \Delta k_y y)dxdy
\]

(7.15)

and longitudinal component for a field, which has a constant transverse distribution within the limits of a crystal, is:
7. SPONTANEOUS DOWN CONVERSION

\[ f_z(\Delta k_z) = F.T.(1) = \int_0^l e^{i\Delta k_z z} dz = i(1 - e^{i\Delta k_z l}) / \Delta k_z \] (7.16)

For a Gaussian pump beam the envelope \( E_{p}(x,y) \) is given by Eq. (7.1) for \( m=0 \),
\[ E_{p}(x,y) = E_{m=0}(r,\phi) = E_0\exp(-r^2 / w_0^2). \] The integration of Eq. (7.15) results in (see Appendix A):
\[ f_{xy,m=0}(\Delta k_x,\Delta k_y) = f_{xy,m=0}(\Delta k_\perp,\theta) = A_{kg} \exp[-(\pi \Delta k_\perp w_0)^2] \] (7.17)
for a vortex pump field the envelope is given by Eq. (7.1) for \( m=1 \),
\[ E_{p}(x,y) = E_{m=1}(r,\phi) = E_0 |r / w_0| \exp(-r^2 / w_0^2)\exp(i\phi), \] and Fourier transformation results in:
\[ f_{xy,m=1}(\Delta k_x,\Delta k_y) = f_{xy}(\Delta k_\perp,\theta) = A_k \Delta k_\perp \exp[-(\pi \Delta k_\perp w_0)^2]\exp(i\theta) \] (7.18)
where \( A_g, A_s, A_{kg}, A_{kv} \) are normalization constants, and \( \Delta k_\perp = \sqrt{\Delta k_x^2 + \Delta k_y^2}, \)
\( \theta = \arctan(\Delta k_y / \Delta k_x) \) are transverse coordinates in \( \Delta k \) space.

7.6 Numerical Investigation

Numerical calculations of the down-conversion signal and idler distributions require the knowledge of properties of the non-linear crystal, which is used for conversion. We modeled parametric scattering in BBO crystal with type-II phase-matching, which was also used in our experimental efforts. The following constants were used: pump wavelength \( \lambda_p = 364 \text{nm} \), signal and idler wavelengths \( \lambda_s = \lambda_i = 728 \text{nm} \), beam size \( w_0 = 1 \text{mm} \), crystal length \( l_c = 7 \text{mm} \) and the incidence angle (angle between the beam and the crystal optical axis) \( \theta = 48^0 \). Calculations followed Eq. (7.9-7.18) and were performed on
Alfa Unix station (21164 processor, 533mhz), numerical grid of 20000x20000 points. Both the signal and idler wave intensity distributions in $k$-space formed a ring. Part of the signal wave ring, calculated using wave-picture model, is shown on Fig. 7.4 for Gaussian and for vortex ($m = 1$) pump beam envelopes. Signal distributions, calculated using quantum-picture model, are similar in appearance. For comparison purposes, the experimentally obtained distribution for a vortex pump beam$^{178}$ is shown on Fig. 7.5. The experimental parameters were the same as the above-mentioned numerical simulation constants, with the exception of spectral distribution, which was defined by 20nm band-pass interference filter, placed in front of the registering digital camera. On Fig. 7.5 both signal and idler rings are shown. We note that signal and idler waves in type-II phase-matching have orthogonal polarizations and may be easily distinguished by placing a linear polarizer in a beam. Experimental results were found to be in qualitative agreement with both numerical models.

The difference between results, numerically calculated with two models, appear to be in the down-conversion ring profiles. Cross-sections of intensity distributions for a Gaussian and vortex pump beam profiles, calculated with a quantum-model, are shown on Fig. 7.6. We note absence of the appreciable difference between distributions. Therefore one could claim that the vorticity and OAM are destroyed in the process of down-conversion. Early experiments with down-conversion of vortices, where down-conversion rings, similar to those shown on Fig. 7.6, were obtained and analyzed, resulted in just such claims$^{193}$. As we noted earlier, the phase information, contained in $k$-space distribution of the pump wave, is indeed removed in the quantum-picture model, and therefore that result should not come as a surprise. However, the wave-model
profiles, shown on Fig. 7.7, indicate that the ring profiles for Gaussian and vortex pump beam may be expected to differ. In the Gaussian case our wave-model produced a *sinc*-function-like profile distribution, in agreement with earlier theoretical results\textsuperscript{121} and our quantum-model. However in the case of a vortex pump beam, the signal ring profile does not have characteristic zeros of a *sinc*-function (see Fig. 7.7). The difference in down-conversion ring profiles for different pump beam envelopes may indicate conservation of OAM in a beam.

The down-conversion ring radius varies with the wavelength of signal wave and therefore experimental observation of exact ring profile would require a bandwidth, approaching a $\delta$-function distribution, as well as a good resolution of the ring structure. In the experimental distribution, shown on Fig. 7.5, neither of these requirements could be satisfied. However experimental confirmation of the conservation of OAM in down-conversion of vortex beam was recently demonstrated by Mair at al.\textsuperscript{179}, using photon-correlation counting technique, and a capability of single photon sorting using interferometers is available\textsuperscript{191}.

**7.7 Conclusion**

We numerically investigated the process of spontaneous down-conversion of optical vortex beams. We derived new phase-matching conditions, which account for the optical vortex OAM. Our modeling, based on new phase-matching conditions, suggested that the vortex in a pump beam might affect the transverse profile of down-conversion rings. Our
results indicated that earlier claims of OAM destruction in the process of down-conversion are unsubstantiated.
Fig. 7.1: Illustration to the notation in Sec. 7.2. Transverse $k$-vector component of a beam is a gradient of the phase $\phi$. 
Fig. 7.2: Spontaneous down-conversion in a non-linear crystal is a process of spontaneous conversion of a pump photon into signal and idler photons.
Fig. 7.3: Profile of the conversion form-factor for a Gaussian and vortex \((m = 1)\) pump beams as function of the transverse radial component \(\Delta k_{\perp,\rho}\) of wave-vector mismatch.
Fig. 7.4: Part of the signal down-conversion ring, numerically calculated using wave-picture. (a) Gaussian pump \((m = 0)\); (b) Laguerre-Gaussian vortex pump beam \((m = 1)\).
Fig. 7.5: Experimentally obtained spontaneous down-conversion signal and idler rings.
Fig. 7.6: Down-conversion signal ring intensity as a function of the transverse radial $k$-vector component, quantum picture. Intensity is shown relative to the maximum intensity of a Gaussian beam. Numerical data points are fitted with smooth curve profiles.
Fig. 7.7: Same as Figure 7.6, wave picture
APPENDIX
A. FOURIER TRANSFORM OF LAGUERRE-GAUSSIAN BEAMS

This Appendix contains mathematical derivation of Fourier-space representations of Laguerre-Gaussian beams. The main result of this derivation is Fourier transform for a truncated beam and therefore the main benefactor of this appendix is Ch. 3.

1. Fourier transform of a Laguerre-Gaussian beam

A Laguerre-Gaussian beam in a transverse plane may be expressed as:

\[ E_m(r, \theta) = E_0 \left| r / w_0 \right|^m \exp\left(-r^2 / 2w_0^2\right) \exp\left(im\theta\right) \]  \hspace{1cm} (A.1)

where \((r, \theta)\) is a set of polar coordinates. In terms of Cartesian coordinates \((x, y)\):

\[ E_m(x, y) = E_0 ([x + iy] / w_0)^m \exp\left(-[x^2 + y^2] / 2w_0^2\right) \]  \hspace{1cm} (A.2)

The far-field (\(k\)-space) distribution of \(E_m(x, y)\) is given by a Fourier transform:

\[ E^\infty(k_x, k_y) = \frac{1}{w_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x, y) \exp(-ik_x x - ik_y y) dx dy \]  \hspace{1cm} (A.3)

where \((k_x, k_y)\) are Cartesian coordinates in \(k\)-space. For simplicity we may express far-field distributions of Laguerre-Gaussian fields in terms of dimension-less coordinates \(t_x = (x / w_0)\), \(t_y = (y / w_0)\) and \(p_x = (k_x w_0)\), \(p_y = (k_y w_0)\). In terms of \((t_x, t_y)\) and \((p_x, p_y)\) Eq. (A.3) becomes:
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\[ E^\omega(p_x, p_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(t_x, t_y) \exp(-ip_xt_x - ip_y t_y) dt_x dt_y \]  
\[ (A.4) \]

For a Gaussian field \((m=0)\) Eq. (A.2) in \((t_x, t_y)\) coordinates becomes:

\[ E_{m=0}(t_x, t_y) = E_0 \exp([-t_x^2 + t_y^2]) \]  
\[ (A.5) \]

The Fourier transform of Eq. (A.5) may be expressed as:

\[ E_{m=0}^\omega(p_x, p_y) = E_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp([-t_x^2 + t_y^2]) \exp(-ip_xt_x - ip_y t_y) dt_x dt_y \]

or, if we separate the variables:

\[ E_{m=0}^\omega(p_x, p_y) = E_0 \int_{-\infty}^{\infty} \exp(-t_x^2) \exp(-ip_xt_x) dt_x \int_{-\infty}^{\infty} \exp(-t_y^2) \exp(-ip_y t_y) dt_y \]

we make use of a table integral\(^{183}\), \(\int_{-\infty}^{\infty} \exp(-t^2) dt = \sqrt{\pi}\) and obtain:

\[ \int_{-\infty}^{\infty} \exp(-t_x^2) \exp(-ip_xt_x) dt_x = \exp(-p_x^2 / 4) \int_{-\infty}^{\infty} \exp(-t_x^2 - ip_xt_x - i^2 p_x^2 / 4) dt_x = \]

\[ \exp(-p_x^2 / 4) \int_{-\infty}^{\infty} \exp([-t_x^2 + i p_x^2 / 2]) dt_x = \pi^{1/2} \exp(-p_x^2 / 4) \]

and, similarly, for \(y\)-component:

\[ \int_{-\infty}^{\infty} \exp(-t_y^2) \exp(-ip_y t_y) dt_y = \pi^{1/2} \exp(-p_y^2 / 4) \]

The resulting Fourier transform of Eq. (A.5) now simplifies to a Gaussian distribution:

\[ E_{m=0}^\omega(p_x, p_y) = E_0 \pi \exp(-p^2 / 4) \]  
\[ (A.6) \]

where \(p^2 = p_x^2 + p_y^2\). We may notice that

\[ E_m(p_x, p_y) = E_0 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( [i \partial / \partial p_x - \partial / \partial p_y]^m \right) \exp([-t_x^2 + t_y^2]) \exp(-ip_xt_x - ip_y t_y) dt_x dt_y \]
and express the higher order Laguerre-Gaussian modes ($m>0$) from $m=0$ mode by differentiating with respect to $p_x$ and $p_y$:

$$E_m^\infty(p_x, p_y) = \left((i \hat{\partial}/\partial p_x - \hat{\partial}/\partial p_y)\right)^m E_{m=0}^\infty(p_x, p_y) \quad (A.7)$$

For example, for $m=1$ we obtain from Eq.’s (A.6) and (A.7):

$$E_{m=1}^\infty(p_x, p_y) = E_0(\pi/2)(-i p_x + p_y) \exp(-p^2/4) \quad (A.8)$$

2. Fourier transform of a truncated Laguerre-Gaussian beams

For a half-beam, truncated by the plane ($x<0$), the field in the near-field is zero for $x<0$ and is given by Eq. (A.2) for $x>0$. The Fourier-transform may be expressed as:

$$E_{H,m}^\infty(p_x, p_y) = E_0 \int_0^\infty \int_{-\infty}^{\infty} \left([t_x + it_y]\right)^m \exp(-[t_x^2 + t_y^2]) \exp(-ip_x t_x - ip_y t_y) dt_x dt_y \quad (A.9)$$

For a Gaussian beam ($m=0$) we again separate the variables:

$$E_{H,m=0}^\infty(p_x, p_y) = E_0 \int_0^\infty \int_{-\infty}^{\infty} \exp(-t_x^2) \exp(-ip_x t_x) dt_x \int_{-\infty}^{\infty} \exp(-t_y^2) \exp(-ip_y t_y) dt_y$$

and for the $x$-coordinate (dimension-less coordinate $t_x$) obtain:

$$\int_0^\infty \exp(-t_x^2) \exp(-ip_x t_x) dt_x = \int_0^\infty \exp(-t_x^2) \left[\cos(p_x t_x) - i \sin(p_x t_x)\right] dt_x =$$

$$(\pi^{1/2}/2) \exp(-p_x^2/4) - i(p_x/2) \exp(-p_x^2/4) \text{F}_1(1/2;3/2;p_x^2/4)$$

where $\text{F}_1(\alpha;\gamma;z)$ is the degenerate hypergeometric function. Integration over the $y$-coordinate was conducted earlier in the derivation of Eq. (A.6). The resulting expression for the $k$-space representation of a half-Gaussian field is given by:

$$E_{H,m=0}^\infty(p_x, p_y) = E_0(\pi/2) \exp(-[p_x^2 + p_y^2]/4)\{1 - i(p_x/2) \text{F}_1(1/2;3/2;p_x^2/4)\} \quad (A.10)$$
For a truncated Laguerre-Gaussian beam of arbitrary \( m \) we may again use the method of derivatives, expressed by Eq. (A.7). For \( m=1 \) we obtain from Eq. (A.10):

\[
E_{H,m=1}^{\infty}(p_x, p_y) = E_0(\pi / 2) \exp(-p^2 / 4) \\
[(-ip_x / 2)(1 - ip_x / 2), F_i(1/2;3/2; p^2_x / 4) + (k_x / 2)(1 - ip_x / 2), F_i(1/2;3/2; p^2_x / 4) \\
+ (1/2), F_i(1/2;3/2; p^2_x / 4) + i(1 - ip_x / 2), \partial F_i(1/2;3/2; p^2_x / 4)]/\partial p_x]
\]

Using relations (see 183, 9.213) \( d_F(\alpha, \gamma, z)/dz = (\alpha / \gamma)F(\alpha + 1, \gamma + 1, z) \),

\[
d(\tau^2)/dt = 2t(df(p)/dp)|_{p=\tau^2}, and an algebraic simplifications, we find:
\]

\[
E_{H,m=1}^{\infty}(p_x, p_y) = E_0(\pi / 4) \exp(-[p_x^2 + p_y^2]/4) \\
[\{1 - i(p_x + ip_y)(1 - ip_x / 2), F_i(1/2;3/2; p^2_x / 4) \\
+ p_x(i + p_x / 2)/3, F_i(3/2;5/2; p^2_x / 4)]
\] (A.11)

For \( m=0, 1 \) hypergeometric functions may be conveniently replaced by Erf functions 183.

In that notation Eq. (A.10) simplifies to

\[
E_{H,m=0}^{\infty}(p_x, p_y) = E_0(\pi / 2) \exp(-p^2 / 4) \\
(\text{A.12})
\]

where Erfc(z) is a complementary error function. For \( m=1 \) Eq. (A.11) becomes:

\[
E_{H,m=0}^{\infty}(p_x, p_y) = E_0(\pi / 2) \exp(-p^2 / 4) \\
(\text{A.13})
\]
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186. To alleviate the problem of diverging in our numerical calculations when the intensity vanishes, we truncated to zero when the intensity fell below a 2% threshold.


