Chapter 3. Wave Propagation through Random Media

3.1 Characteristics of Wave Behavior

Sound propagation through random media is the central part of this investigation. This chapter presents a frame of reference from which to understand the underlying acoustic principles of sound waves as they pertain to the measurements reported herein.

Any assembly of particles which mutually interact is a medium in which wave motion may occur. A brief disturbance in a small region induces motion in neighboring regions, resulting in some sort of movement which eventually spreads throughout the medium. This transmission of disturbances may be called wave propagation (Baldock, 1981). Consider a scalar physical quantity $u$ which is dependent on a single space coordinate $x$ and on a given time $t$. If, in some domain of $xt$-space, $u$ can be expressed as:

$$ u(x,t) = f(x - ct) $$

(3.1.1)

where $c$ is constant, then $u$ is said to be a wave which propagates in an $x$-direction with velocity $c$ (Baldock, 1981). Thus, every wave traveling with speed $c$ in any $x$-direction satisfies the one-dimensional classical wave equation:

$$ c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} $$

(3.1.2)
Likewise, a function $u(r, t)$ defined on a region of three dimensional space over some time interval is a plane wave moving in the positive $x$-direction by equation (3.1.1). On any plane perpendicular to the $x$-axis, $u$ is constant, and the plane moves in a positive direction with constant velocity $c$. Hence, we can define this plane as the wave front. A plane wave moving in the direction of a unit vector $n$ is defined by:

$$u = f(r \cdot n - ct) \quad (3.1.3)$$

From equation (3.1.3) we can derive the three-dimensional classical wave equation defined as:

$$c^2 \nabla^2 u = \frac{\partial^2 u}{\partial t^2} \quad (3.1.4)$$

satisfied by all plane waves traveling with the same speed $c$ (Baldock, 1981).

As outlined by Tatarskii (1961) in his discussion of sound wave scattering in a locally isotropic turbulent flow, the movement of sound waves through a turbulent flow stream is analogous to the phenomenon of electromagnetic wave scattering. The overall velocity of a propagating wave is influenced by the relative fluid velocity and surrounding temperatures. Temperature fluctuations are important as they cause fluctuations in the sound speed. The speed of sound can be expressed as:

$$c^2 = \gamma p / \rho = \gamma RT \quad (3.1.5)$$

for perfect gases,

$$\gamma = (R + c_v) / c_v \quad (3.1.6)$$

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As shown by Eq. (3.1.5) above, the square of the sound speed is proportional to gamma, \( \gamma \) which ranges from a maximum of \( 5/8 \) for monatomic gases through \( 7/5 \) for diatomic gases, \( R \) the relative gas constant, and \( T \) the temperature.

Tatarskii (1961) gives the basic equation for sound propagation in a moving random medium to be written in the form:

\[
\Delta P = \frac{1}{c^2} \left( \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} \right)^2 P = 0
\]  

(3.1.7)

where \( P \) denotes the sound wave potential, \( u_i \) the components of the velocity motion of the described medium, and \( c \) the relative sound speed. More specific to our investigation, Andreeva (2003) outlines, at length, the applicability of two well known approximate theories of wave propagation, ray acoustics and the Rytov method.

### 3.2 Ray Theory and Statistics

Chernov (1960) described acoustic wave propagation in a medium with random inhomogeneities by ray theory and statistics. The theory is applicable provided that \( a \), the scale of the inhomogeneities, is large compared to the wavelength \( \lambda \). Thus, this condition is often satisfied for ultrasonic waves (Chernov, 1960). If this condition is satisfied, the application of ray theory is valid in regions of linear dimension \( L \), where \( L \) satisfies the condition \( \sqrt{aL} \ll a \). However, at larger propagation distances the theory breaks down, and warrants the use of diffraction theory. Therefore assuming the following:
\[ \lambda \ll a \quad (3.2.1) \]
\[ \sqrt{\lambda L} \ll a \quad (3.2.2) \]

and that the transit time of the ray is small compared to the characteristic scale of changes of the inhomogeneities in time, the ray equation defined in Eq. (3.2.3) can be obtained from Fermat’s principle.

\[ \int_{A}^{B} \frac{d\sigma}{c} = \min \quad (3.2.3) \]

We can define the refractive index by:

\[ n = \frac{c_0}{c} \quad (3.2.4) \]

Thus, Eq. (3.2.3) can be re-written in the form below:

\[ \int_{A}^{B} n(x, y, z) d\sigma = \min \quad (3.2.5) \]

Furthermore, we assume that collectively, the ray trajectories are described by a family of curves expressed by the equations \( x = x(u) \), \( y = y(u) \), and \( z = z(u) \) that pass through given points A and B (Chernov, 1060).

The time taken for a ray to travel a given distance changes with varying conditions in the medium. This results in ray tube deformation, and subsequently intensity fluctuations (Chernov, 1960). Assuming that deviation of the rays from their
initial direction with respect of the x-axis is small, the travel time and mean travel time of a propagating wave over a distance $L$ is given by:

$$ t = \frac{1}{c_0} \int_0^L n(x, y, z) dx $$

(3.2.6)

$$ \bar{t} = \frac{1}{c_0} \int_0^L n(x, y, z) dx = \frac{1}{c_0} \int dx $$

(3.2.7)

where the corresponding values of the refractive index $n(x, y, z)$ are taken along the ray.

Since $\bar{n} = 1$, the deviation from the mean time can be expressed by Eq. (3.2.8) as follows (Chernov, 1960).

$$ \Delta t = t - \bar{t} = \frac{1}{c_0} \int n(x, y, z) dx - \frac{1}{c_0} \int dx = \frac{1}{c_0} \int \mu(x, y, z) dx $$

(3.2.8)

By assuming that the values of $x$ in the correlation coefficient $N(x, y, z)$ are of the order $a(x \sim a)$, hence the mean square transit time fluctuations are defined by Eq. (3.2.9) (Chernov, 1960).

$$ \overline{\Delta t^2} = \frac{2\mu^2 L}{c_0^2} \int N(x, 0, 0) dx $$

(3.2.9)

Thus, we can calculate the phase fluctuations $S' = \omega \Delta t$ by Eq. (3.2.10), and subsequently the relative change in intensity along a path $dx$ by Eq. (3.2.11) below:
\begin{equation}
S^{r2} = 2\mu^2 k^2 L \int_0^\infty N(x,0,0)dx
\tag{3.2.10}
\end{equation}

\begin{equation}
\frac{dl}{l} = -dx \int \nabla^2 \mu dx'
\tag{3.2.11}
\end{equation}

where $k$ is the wavenumber (Chernov, 1960).

Subsequently these methods, based in ray theory, are generally used to describe the propagation of sound in a slowly changing medium state (Andreeva, 2003). The grid-generated turbulence in our experiment, to a large degree, is comparable to the turbulent atmosphere, featuring fluctuations in flow velocity and temperature. Equations (3.2.1) and (3.2.2) were used to determine if the conditions of the ray approximation is satisfied with respect to our investigation. The wavelength was determined by $\lambda = \frac{v}{f}$, where $v$ is the propagation speed and $f$ the respective frequency. Thus, for our 100 kHz transducers the wavelength was estimated to be approximately 3mm. The turbulent length scales as outlined in section 2.1 are not appreciably larger than the wavelength, and the left side of equation (3.2.2) as required. As such, the effect of these flow fluctuations on sound propagation cannot be treated by the methods of ray acoustics (Andreeva, 2003). Rather, a combination of the statistical representation of isotropic and homogenous turbulence and the Kolmogorov (1941) “2/3” law provide a foundation for the theory of wave propagation in turbulent media.
3.3 Kolmogorov’s “2/3” Law and Resulting Travel Time Equations

Kolmogorov’s “2/3” law states that the velocity fluctuations at two different points in the flow stream are proportional to the distance between these points raised to the 2/3 power (Frisch, 1995). Sound wave propagation between two transducers over a distance $L$, in a direction $t_1$ and opposite direction $t_2$, can be described by a derivation of the flowmeter equation (Andreeva, 2003). Accordingly, by Andreeva (2003) we can express the acoustic travel times of these waves as:

$$
\int_0^L \frac{dy}{c - u_1} \approx t_0 + \frac{1}{c^2} \int_0^L u_1 dy 
$$ (3.3.1)

$$
\int_0^L \frac{dy}{c + u_2} \approx t_0 - \frac{1}{c^2} \int_0^L u_2 dy 
$$ (3.3.2)

where $t_0$ denotes the travel time in the undisturbed media, and $c$ the sound speed. The velocities $u_1$ and $u_2$ are defined by:

$$
u_1 = U \sin \beta + u_1'$$ (3.3.3)

$$
u_2 = U \sin \beta + u_2'$$ (3.3.4)

where $U$ is the mean flow velocity, and $u_1'$, $u_2'$ are the respective fluctuations along the sound path. Neglecting terms of order $U / c$, $U^2 / c^2$, the time difference is expressed by Eq. (3.3.5) as follows:
\[
\Delta t = t_2 + t_1 - 2t_0 \approx \frac{1}{c^2} \int_0^L (u'_1 - u'_2) dy = \frac{1}{c^2} \int_0^L \Delta u dy
\] (3.3.5)

The turbulent velocity fluctuations in Eq. (3.3.5) which influence the acoustic travel time can be described by a random function of time and position. Thus, Eq. (3.3.5) is rewritten as:

\[
\overline{\Delta t^2} = \frac{1}{c^4} \int_0^L dy' \int_0^L dy'' \Delta u(y') \Delta u(y'')
\] (3.3.6)

Invoking the Kolmogorov “2/3” law, we account for the correlation of fluctuations at different points in the flow, and so (Andreeva, 2003):

\[
[u(y') - u(y'')]^2 = C^2 R^{2/3}
\] (3.3.7)

where \( R \) is defined by the distance between the points \( y' \) and \( y'' \), and \( C \) a constant characteristic of turbulence having dimensions of \( cm^{2/3} \cdot s^{-1} \). Under the isotropic and homogenous turbulent assumption, the left hand side of Eq. (3.3.7) is expressed by Eq. (3.3.8).

\[
C^2 R^{2/3} = \overline{[u(y') - u(y'')]^2} = 2[\overline{u(y')^2} - u(y')u(y'')]
\] (3.3.8)

Thus, from transducer geometry we can define:

\[
R^2_1 = \delta^2 + (y' - y'')^2
\] (3.3.9)

\[
R^2_2 = (y'' - y')^2
\] (3.3.10)
By way of Eq. (3.3.9) and Eq. (3.3.10) and use of the “2/3” law, we rewrite the integrand of Eq. (3.3.6) as follows:

$$
\Delta u(y') \Delta u(y'') = C^2 \left\{ R_1^{2/3} - R_2^{2/3} \right\} = -0.5C^2 \left\{ 2(y' - y'')^{2/3} - 2\left(y' - y''\right)^2 + \delta^2 \right\}^{1/3}
$$  \hspace{1cm} (3.3.11)

Substituting Eq. (3.3.11) into Eq. (3.3.6) we define the expression:

$$
\overline{\Delta t^2} = C^2 L \left( \frac{1}{c^2} \right)^2 \delta^{5/3} \text{const}
$$  \hspace{1cm} (3.3.12)

where the constant is determined experimentally by Obukhov (1951) to be equal to 3 (Andreeva, 2003).

### 3.4 Fermat’s Principle

Fermat’s principle states that the path of a ray of light between two points is the path that minimizes the travel time (for which it is an extremum). For example, let the ray traveling from a source in medium $v_1$ located at (0, a) to a receiver in medium $v_2$ at (b, -c), pass through the interface between the two media at some point (x, 0). To completely determine the ray path, we must find $x$ such that the total travel-time $T(x)$ for the path defined by $x$ is an extremum.
The time necessary for the ray to travel from (0, a) to (b, -c) via (x, 0) is defined by:

\[
T(x) = \frac{(a^2 + x^2)^{1/2}}{v_1} + \frac{(b - x)^2 + c^2)^{1/2}}{v_2}
\]  
\(3.4.1\)

Thus, to define x such that T(x) is an extremum, we set the first derivative to zero:

\[
0 = \frac{dT(x)}{dx} = \frac{x}{v_1(a^2 + x^2)^{1/2}} - \frac{b - x}{v_2[(b - x)^2 + c^2)^{1/2}] = \frac{\sin i_1}{v_1} - \frac{\sin i_2}{v_2}
\]  
\(3.4.2\)

Hence, we arrive at Snell's Law:

\[
\frac{\sin i_1}{v_1} = \frac{\sin i_2}{v_2}
\]  
\(3.4.3\)
Thus, so that the refracted ray path is actually a minimum time path we note that:

\[
\frac{d^2 T(x)}{dx^2} = \frac{a^2}{v_1(a^2 + x^2)^{1/2}} + \frac{c^2}{v_2[(b-x)^2 + c^2]^{3/2}} > 0 \tag{3.4.4}
\]

This relationship is also true for sound waves propagating between two different medium.

With regard to our experiment, the principle states that the travel path between the ultrasonic transmitter and receiver always is an extremum. Hence, the length of the travel path, measured along a ray, is always shorter then any other path (Andreeeva, 2003).

### 3.5 Caustics

The basic equations and methods of ray theory which describe wave motion throughout random media may become invalid as rays come within close proximity to each other to form caustics. The singularities of geometrical optics are caustics, and their systematization by catastrophe theory is perhaps the most significant application of that theory to date (Nye, 1999). The diffraction patterns associated with the caustics are dominated by wave dislocations, which are line singularities of the phase, analogous to crystal dislocations. A polarized wave field possesses an even finer structure of singularities (Nye, 1999). According to Lighthill (1978), a caustic is defined as a boundary between a region with a complicated wave pattern, due to interference between two groups of waves. In terms of ray theory, by Spetzler and Snieder (2001), the concept of caustics is understood as the focus point in space through which rays go; whereas the
consequence of their production in a wavefield is that the amplitude is infinitely high at
the focus point because the geometrical spreading factor is zero at the caustic point.

The caustic phenomenon has been the focus of several investigations, such as
those conducted by White et al. (1988), where he used limit theorems for stochastic
differential equations on the equation of dynamic ray tracing to predict when caustics
start to develop in Gaussian random media (Spetzler, 2001; Snieder, 2001). In addition,
used Chapman's method to write explicitly the variation of 2-D and 3-D wavefields in the
vicinity of focus points (Spetzler, 2001; Snieder, 2001). Lighthill (1978) uses the Airy
integral to address the contradictions of a straightforward ray acoustics theory and
account for the local singularities in the presence of the caustics. However, for our
purposes we will narrow the scope of our interest to the behavior of propagating waves
before they strongly interact to form caustics, but, interact strongly enough for their
influence to be seen in the non-linearity of the amplitude variance.