Option Pricing Using Monte Carlo Methods

A Directed Research Project

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Financial Mathematics

by

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Approved:

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Abstract

This project is devoted primarily to the use of Monte Carlo methods to simulate stock prices in order to price European call options using control variates, and to the use of the binominal model to price American put options. At the end, we can use the information to form a portfolio position using an Interactive Brokers paper trading account. This project was done as a part of the masters capstone course Math 573: Computational Methods of Financial Mathematics.
Acknowledgements

This project could not have been completed without the support of Professor Marcel Blais, whose support and enthusiasm throughout the process of finishing this project and profession made this a meaningful exercise.
# Table of Contents

1. Introduction .................................................................................................................. 7

2. Process .......................................................................................................................... 8
   2.1. Selection of Assets ................................................................................................. 8
   2.2. Parameter .............................................................................................................. 8
   2.3. Simulation of Stock Price ...................................................................................... 10
   2.4. Price European Call Option .................................................................................. 11
   2.5. Price American Put Option ................................................................................... 12

3. Performance .................................................................................................................. 15

4. Conclusion .................................................................................................................... 15

5. References and Data .................................................................................................... 16

6. Appendix ...................................................................................................................... 17
List of Tables

Table 1 Initial Stocks ........................................................................................................ 8
Table 2 Final Call option positions ............................................................................... 12
Table 3 Final Put option positions ............................................................................... 13
List of Charts

Table 1 Portfolio Performance........................................................................................................ 14
Table 2 AMJ May 20’11 38 Put Option ...................................................................................... 14
Table 3 GM May 20’11 30 Call Option.......................................................................................... 15
1. Introduction

The purpose of this project is to use Monte Carlo methods to price European Call options on equities and to use the binominal model to price American put options. The portfolio positions are formed in an interactive Brokers paper trading account\(^1\) using the information obtained through the pricing process.

As the first step I choose ten stocks with historical data\(^2\) attached and ten options. Second, I calculate the expected daily return for each stock and covariance matrix of daily returns. Third, based on the historical data I use a multidimensional geometric Brownian motion to simulate stock prices.

At last, but not least, in order to obtain option prices, I apply the variance reduction method of control variates for European call options, while implementing binominal model for American put options.

Finally, I compare the numerical prices with actual prices to make investment decisions.

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\(^1\) Interactive Brokers paper trading account is from [http://www.interactivebrokers.com](http://www.interactivebrokers.com)

\(^2\) Data is downloaded from [http://finance.yahoo.com](http://finance.yahoo.com)
2. Process

2.1. Selection of Assets

I divide my portfolio into four main sections: financial services, Internet technology, motor industry, and retail sales.

Table 1

<table>
<thead>
<tr>
<th>Financial Services</th>
<th>Internet Technology</th>
<th>Motor Industry</th>
<th>Retail sales</th>
</tr>
</thead>
<tbody>
<tr>
<td>AMJ</td>
<td>MSFT</td>
<td>HMC</td>
<td>WMT</td>
</tr>
<tr>
<td>JPM</td>
<td>ORCL</td>
<td>GM</td>
<td>TESO</td>
</tr>
<tr>
<td>BAC</td>
<td>IBM</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

I chose those companies by using some of the following criteria:

1. The company has a promising future.
2. The stock price of the company fluctuates slightly.
3. The company is one of the leading companies in its industry.
4. The option has a wide bid-ask spread.

2.2. Parameter

2.2.1. Stock price

The stock price is represented by capital S. $S_{i,t}$ means the $i^{th}$ stock price at time $t$. ($0 <= t <= N$).

$N$ is the number of days that stocks traded from 1/1/2011 to 4/22/2011.

$i=1,2,3, \ldots, d; \ d \ is \ the \ total \ number \ of \ stocks \ (in \ this \ case, \ d=10) \ t=1,2,3,\ldots, N$

---

3 The data is downloaded from [http://finance.yahoo.com/](http://finance.yahoo.com/)
2.2.2. Daily Return

\[ r_{i,t} = \frac{S_{i,t} - S_{i,t-1}}{S_{i,t-1}} \]

2.2.3. Expected Return Vector

\[ E(r_{i,t}) = \frac{\sum_t(r_{i,t})}{N} \]

2.2.4. Covariance Matrix

\[ \text{cov}(r_i, r_j) = E[(r_i - E[r_i])(r_j - E[r_j])] \]

where

N is the number of days that stocks traded from 1/1/2011 to 4/22/2011.

i=1,2,3,...,d. j=1,2,3,...,d. d is the total number of stocks (in this case, d=10)

N=1,2,3,...,N

2.2.5. Risk Free Rate

The risk free rate in this project is the average of the three month Federal Interest Rate.\(^4\)

2.2.6. Cholesky factorization of \(\Sigma\)

A is obtained from the Cholesky factorization of \(\Sigma\) \(^5\)

---


\(^5\) Monte Carlo Methods in Financial Engineering – Paul Glasserman
2.3. Simulation of Stock Price

The first step uses multiple dimensional geometric Brownian motion to simulate the stock prices.

Algorithm:

\[ S_i(t_{k+1}) = S_i(t_k) e^{(\mu_i - 0.5 \sigma_i^2 \Delta t_i) + \sqrt{\Delta t_i} \sum_{j=1}^{d} A_{ij} Z_{k+1,j}} \]

\( i \) is the index of stocks

\( k \) is the index of steps

\( Z_k = (Z_{k1}, Z_{k2}, \ldots, Z_{kd}) \sim N(0, I) \), \( Z_1, Z_2, \ldots, Z_n \) are independent

\( d \) is the total number of stocks

\( n \) is the total number of steps.

\( A \) is obtained from Cholesky factorization of \( \Sigma \).
2.4. Price European Call Options

After obtaining the simulated prices of the targeted stocks, I can calculate the European call option price with various strike prices.

First of all, in order to estimate the option prices on May 20, 2011, I set the stock prices at April 22, 2011 as initial stock prices. Secondly, I count the number of days until maturity in terms of year. Thus, $T = 0.1111(28/252)$. Using the same method we recalculate the maturity $T$, I transform the covariance matrix of the daily returns to the one of yearly returns. ($\Sigma = \text{covariance matrix}/\sqrt{252}$)

For European call option, the payoff is:

$$P = \max(S_T - K, 0)$$

The second step uses a variance reduction method to control variance of option prices.

Algorithm:

$$E(\bar{Y}(b)) = E(\bar{Y} - b(\bar{X} - E(X))) \quad 7$$

Where

$$b = \frac{cov(X, Y)}{Var(X)}$$

$\bar{Y}$ contains the mean value of discounted payoffs which are simulated using multiple dimensional geometric Brownian motion. $\bar{X}$ is a control variate estimator (stock prices in this case). $X$ includes the stock prices simulated using a multiple dimensional geometric Brownian motion. \(^7\)

According to simulation process mentioned above, I have obtained the results below:

\(^7\) Monte Carlo Methods in Financial Engineering --Paul Glasserman
2.5. Pricing American Put Options

Comparing to the valuation of a European option, the valuation of an American option is a difficult problem in pricing because it involves the determination of optimal exercise timing due to the fact that the option can be exercised at any time prior to its own maturity. The option holder will face a choice of either exercising the option immediately or holding the option for a better position. Therefore in order to produce the optimal value at current point in time, we have to compare the exercising payoff value with the value of holding the option.

The parameters are obtained as following:

\[ u = \exp((r - 0.5\sigma^2)\Delta t + \sigma \sqrt{\Delta t}) \]
\[ d = \exp((r - 0.5\sigma^2)\Delta t - \sigma \sqrt{\Delta t}) \]
\[ p_u = 0.5 \]
The option price will be

\[ V_{j,l} = \max\{\max(K - S_{j,l}, 0), \exp(-r\Delta t) \left(p_u V_{j+1,l+1} + (1 - p_u)V_{j+1,l}\right)\} \]

\(^8\)

\(u\) is the upward factor.

\(d\) is the downward factor.

\(p_u\) is the probability of an upward movement.

\(j\) is the time step.

\(I\) is the position variable at \(j^{th}\) time.

\(V\) is the American put option price.

\(K\) is the strike price.

\(S\) is the stock price.

According to the algorithm and equations above, I need to calculate the option price backwards, which means tracking the option price from the maturity date back to the initial date in order to obtain the optimal option price that we seek.\(^8\)

We have the following result:
<table>
<thead>
<tr>
<th></th>
<th>S0 (april.22)</th>
<th>Ans</th>
<th>K</th>
<th>BID</th>
<th>ASK</th>
</tr>
</thead>
<tbody>
<tr>
<td>BAC</td>
<td>14.19</td>
<td>2.8053</td>
<td>17</td>
<td>4.7</td>
<td>4.8</td>
</tr>
<tr>
<td>AMJ</td>
<td>36.48</td>
<td>1.5149</td>
<td>38</td>
<td>1.5</td>
<td>1.6</td>
</tr>
<tr>
<td>JPM</td>
<td>43.58</td>
<td>0.4079</td>
<td>44</td>
<td>0.48</td>
<td>0.5</td>
</tr>
<tr>
<td>MSFT</td>
<td>27.98</td>
<td>2.0138</td>
<td>30</td>
<td>3.8</td>
<td>4.2</td>
</tr>
<tr>
<td>ORCL</td>
<td>31.62</td>
<td>3.3709</td>
<td>35</td>
<td>3.2</td>
<td>3.3</td>
</tr>
<tr>
<td>IBM</td>
<td>147.48</td>
<td>12.4884</td>
<td>160</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>HMC</td>
<td>40.02</td>
<td>4.9657</td>
<td>45</td>
<td>7</td>
<td>7.8</td>
</tr>
<tr>
<td>GM</td>
<td>37.06</td>
<td>2.9271</td>
<td>40</td>
<td>8.6</td>
<td>9</td>
</tr>
<tr>
<td>WMT</td>
<td>54.56</td>
<td>0.4307</td>
<td>55</td>
<td>1.75</td>
<td>1.77</td>
</tr>
<tr>
<td>TESO</td>
<td>16.02</td>
<td>1.4712</td>
<td>17.5</td>
<td>1.4</td>
<td>1.6</td>
</tr>
</tbody>
</table>
3. Performance

My portfolio did not perform very well as the following graph illustrates:

![Chart 1](image1.jpg)

![Chart 2](image2.jpg)
4. Conclusion

Comparing the numerical option prices with actual market prices, we find that the numerical option prices do not match very well with the real market prices. This may be due to the fact that the estimation of stocks’ volatility is not accurate.

During the programming process, I encountered several difficulties. First, I was wondering which is the appropriate discount rate between the federal fund interest rate and US Treasury bill rate. Secondly, I had a problem choosing suitable strike prices.

As a result, if I had chance to do this project again, I would like to collect more data in order to make my simulation process more reliable and to choose more reasonable strike prices. Furthermore, in order to fix the inaccurate estimation problem of volatility I mentioned above, I would like to try industry factor model to gauge the covariance matrix instead of calculating it directly.
5. References and Data

References - Literature


Data


6. Appendix:

This is the main code, which includes the call option pricing and American put option pricing codes.

\[
\text{matrix} = \text{csvread('stockprices.csv')};
\]
\[
T = 2 \frac{8}{252}; N = 200; m_t = 100; \quad \% \ T \text{ is the maturity days counted as year}
\]
\[
r = 0.001; \quad \% \text{federal fund rate}
\]
\[
K_{\text{price}} = [13, 35, 42, 27, 31, 145, 40, 37, 50, 15]; \quad \% \text{set strike price}
\]
\[
[n, m] = \text{size(matrix)};
\]
\[
\text{retprice} = \text{zeros}(n-1, m);
\]
\[
% \text{calculate the relative daily return}
\]
\[
\text{for } i = 2:n
\]
\[
\text{retprice}((i-1,:), :) = (\text{matrix}(i,:) - \text{matrix}(i-1,:))./\text{matrix}(i,:);
\]
\[
\text{end};
\]
\[
muret = \text{mean(retprice)};
\]
\[
% \text{make covariance matrix measures the cov as years.}
\]
\[
\text{covmatrix} = \text{cov(retprice)};
\]
\[
\text{covmatrix} = \text{covmatrix}/\sqrt{252};
\]
\[
yhat = \text{zeros}(10, (m_t + 1)); \text{yy} = []; \text{Stm} = \text{zeros}(10, m_t);
\]
\[
% \text{replicate GBM 100 times, get option price=max(st-k,0)}
\]
\[
\text{for } ii = 1:m_t
\]
\[
\text{temp} = \text{zeros}(10, (N + 1)); \text{opt} = \text{zeros}(10, 1);
\]
\[
\text{temp} = \text{multipleGeoBrownianMotion(muret, covmatrix, T, N, matrix(n,:))};
\]
\[
\text{Stm(:,ii)} = \text{temp(:, length(temp))};
\]
\[
\text{opt} = \text{max}((\text{Stm(:,ii)} - K_{\text{price}}'), 0);
\]
\[
y = \text{opt} \ast \text{exp}(-r \ast T);
\]
\[
\text{yy} = [\text{yy}, y];
\]
\[
\text{end};
\]
\[
% \text{use variance reduction method}
\]
\[
\text{for } ii = 1:10
\]
\[
\text{cov22} = \text{cov}((\text{Stm}(ii,:), \text{yy}(ii,:)));
\]
\[
\text{b}(ii) = \text{cov22}(1, 2)/\text{var(Stm(ii,:))};
\]
\[
\text{end};
\]
\[
\text{meanstm} = \text{mean(Stm, 2)};
\]
\[
\text{Stemp} = \text{zeros}(10, m_t);
\]
for ii=1:nt
    Stemp(:,ii)=Stm(:,ii)-meanstm;
end;
b=b';
yhat=[];
for ii=1:10
    yhat(ii,:)=yy(ii,:)-b(ii)*Stemp(ii,:);
end;
mean(yhat,2) % output the option price.

% the following is the american put option

Kprice2=[17,38,44,30,35,160,45,40,55,17.5]; % set strike price for put option
[n,m]=size(matrix);
retprice=zeros(n-1,m);
% calculate the relative daily return
for i=2:n
    retprice((i-1),:)=(matrix(i,:)-matrix(i-1,:))/matrix(i,:);
end;
muret=mean(retprice);
covmatrix=cov(retprice);
outcome=[];
for kk=1:m
    sigma=std(retprice(:,kk)); % get the standard deviation of relative daily return.
deltat=T/N; % dt
u=exp((r-1/2*sigma^2)*deltat+sigma*sqrt(deltat)); % the upward multiplier
d=exp((r-1/2*sigma^2)*deltat-sigma*sqrt(deltat)); % the downward multiplier
p=1/2; % possibility of up and down.
s0=matrix(n,kk); % initial stock price
bioput=[];
strike=Kprice2(kk); % set strike price for certain stock
for i=N:(-1):1 % we backtrack the option from N down to 1
    for j=1:i
        root=s0;
root=root*u^j*d^(i-j);
bioput(i,j)=max(strike-root,0);  % calculate the option price at current step
end;
if i<N
  for j=1:i
    bioput(i,j)=max(bioput(i,j),exp(-
r*T/N)*(0.5*bioput(i+1,j)+0.5*bioput(i+1,j+1)));
    % compare current option price and the discounted, expected option
    % of forward option price, and choose the better one.
  end;
end;
end;
output(kk)=bioput(1,1);
end;
output'

This is the GMB mode needed in the main file.

function [S]=multipleGeoBrownianMotion(mu,Sigma,T,N,S0)
d=size(Sigma,1);
A=chol(Sigma)';
St=zeros(d,N+1);
St(:,1)=S0;
for ii=1:N
  random=randn(d,1);
  for jj=1:d
    St(jj,ii+1)=St(jj,ii)*exp((mu(jj)-
    Sigma(jj,jj)^2/2)*(T/N)+sqrt(T/N)*A(jj,:)*random);
  end;
end;
S=St;