Quantum Phase of a Bose-Einstein Condensate with an Arbitrary Number of Atoms

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Abstract:

This project focused on learning and reproducing aspects of BECs. This was accomplished through background research, calculating its density of energy states when inside a parabolic trap, finding the relationship for the critical BEC temperature for BEC inside a parabolic trap, looking in depth into the paper “Quantum Phase of a Bose-Einstein Condensate with an Arbitrary Number of Atoms” [1], performing MATLAB simulations using the paper’s interference phase building method, and evaluating some simulation results.
Contents
Abstract:........................................................................................................................................... ii
1. Project Goals and Objectives ....................................................................................................... 4
2. Introduction to Bose-Einstein Condensates: .................................................................................. 5
   2.1 Bose-Einstein Condensates ...................................................................................................... 5
   2.2 How do Bose-Einstein Condensates Form? .............................................................................. 6
   2.3 Bose-Einstein Condensate Theory .......................................................................................... 9
      2.3.1 Bose-Einstein Statistics .................................................................................................... 9
      2.3.2 Density of States .............................................................................................................. 10
      2.3.3 Bose-Einstein Critical Temperature ............................................................................... 13
3. “Quantum Phase of a Bose-Einstein Condensate with an Arbitrary Number of Atoms” ........... 16
   3.1 Introduction to Waves and Interference .................................................................................. 16
   3.2 Building an Interference Phase Atom by Atom ...................................................................... 18
   3.3 Simulating the Interference of N Atoms ................................................................................ 24
   3.4 Interpretation of Results ....................................................................................................... 26
4. Conclusions .................................................................................................................................... 32
5. References ...................................................................................................................................... 34
6. Appendixes:................................................................................................................................... 35
   6.1 Substitution for Equation (21) ................................................................................................. 35
   6.2 Matlab Code for Figure 6 ....................................................................................................... 35
   6.3 MATLAB for P(x1,x2) .......................................................................................................... 35
   6.4 MATLAB for Interference of BECs ....................................................................................... 36
   6.5 Quantum Phase of a Bose-Einstein Condensate with an Arbitrary Number of Atoms ........... 38
1. Project Goals and Objectives

One of the goals of this project is to learn and present an understanding of Bose-Einstein Condensates. I accomplished this in chapter two by giving a description of Bose-Einstein Condensates, solving for the density of states $D(E)$ for a parabolic magneto-optical trap (16), finding the critical Bose-Einstein Condensate temperature $T_b$ for a parabolic magneto-optical trap (22), and finding the expected occupancy of the ground (BEC) state as a function of temperature (24) for a parabolic magneto-optical trap which is plotted in Figure 6.

Another goal of this project is to learn and reproduce theory from a paper called “Quantum Phase of a Bose-Einstein Condensate with an Arbitrary Number of Atoms” written by Juha Javanainen and Sung Mi Yoo [1]. I accomplished this in chapter three by reviewing facets of the paper, performing MATLAB simulations using the paper’s interference phase building method, and evaluating some simulation results.
2. **Introduction to Bose-Einstein Condensates:**

Chapter two introduces Bose-Einstein condensates by answering the following questions: What are some properties of Bose-Einstein Condensates, how do Bose-Einstein Condensates form, and what is the theory behind Bose-Einstein Condensate formation?

2.1 **Bose-Einstein Condensates**

BEC is a form of matter that only occurs at temperatures close to absolute zero. Quantum mechanically speaking, BEC is defined as bosons that occupy a single particle state in the ground state (lowest energy level).

![Figure 1](image-url)
Bosons are particles with integer spin that do not obey the Pauli exclusion principle because they can occupy the same state [2] [3]. In Quantum Mechanics, the wave functions of bosons are indistinguishable from one another when their wave functions overlap because the observer does not know if the bosons have switched position after consecutive measurements [3]. Identical bosons tend to be closer together than identical fermions (half integer spin particles that obey the Pauli exclusion principle) because their exchange force, which is not really a force at all but a consequence of the symmetrization requirement

\[
\psi(r_1, r_2) = \pm \psi(r_2, r_1),
\]

where + is for identical bosons and – is for identical fermions [3]. BEC can be thought of as identical bosons behaving as a single particle-wave instead of separate particle-waves because the bosons share the same state [3]. BECs obey Bose-Einstein distributions, equation (5), found later in this chapter. BEC is often compared to a laser because lasers are made of photons, which are also bosons and traveling at the same frequency and phase with one another. Because BEC acts as a single particle-wave, quantum mechanical effects like wave interference become apparent on a larger scale relative to the scale of a single atom. Figure 1 shows interference between two BEC and it is important to note that the BEC does not interfere until laser photons are absorbed, which instantaneously raises the bosons out of the ground state until the emit the photons. The light and dark bands are interference fringes on the BEC.

### 2.2 How do Bose-Einstein Condensates Form?

BEC formation was predicted by Bose and Einstein in 1924-25 using Bose-Einstein Statistics but was not observed until 1995 when Eric Cornell and Carl Wieman first observed BEC at the University of Colorado at Boulder [4]. Cornell, Wieman, and Wolfgang Ketterle from MIT received the 2001 Nobel Prize in Physics for their observation [4]. BEC formation occurs when bosons are cooled to temperatures on the order of nano-Kelvin and the seventy-
year gap between theory and experimentation is due to how difficult it is to cool atoms down to that low of a temperature [4].

To reach micro-Kelvin, using laser cooling of atoms held inside a magneto-optical trap is common. A magneto-optical trap is a device that combines the effects of magnetic trapping and laser cooling [5]. A magnetic trap’s potential is determined by a spatially varying magnetic field produced by running current through Helmholtz coils and the presence of a laser field [6]. In the trap, the most energetic atoms can move farther against the pull of the magnetic forces before they are pulled back to the center of the trap [6]. The Zeeman effect is the shifting of energy levels due to an atom in the presence of an external magnetic field $B_{\text{ext}}$ [3]. Because the energy levels are shifted, the spectral lines are also shifted because the frequency of light that can be emitted or absorbed is proportional to the difference in energy levels. For a hydrogen atom where $B_{\text{ext}} \gg B_{\text{int}}$, the Zeeman Hamiltonian (the difference in an energy level that the magnitude of $B_{\text{ext}}$ exerts) is,

$$H'_Z = - (\mu_l + \mu_s) \cdot B_{\text{ext}},$$

where $\mu_s$ is the magnetic moment associated with the electron spin and $\mu_l$ is the magnetic moment associated with orbital motion [3]. Because $B_{\text{ext}} \gg B_{\text{int}}$ we can approximate the energy levels of hydrogen to be,

$$E_{nm,m_s} = - \frac{13.6 \text{ eV}}{n^2} + \mu_B B_{\text{ext}} \left( m_l + 2m_s \right),$$

which shows that the energy level has split because the associated electron spin can have either values $\pm \frac{1}{2}$ [3].

Laser cooling is a method that takes advantage of the Doppler shift to cool atoms and has to take into account the Zeeman effect to find the most effective frequency of the lasers to use [6]. The lasers used in laser cooling are set to a frequency slightly below the average absorption frequency of the atoms so then on average the atoms will emit a frequency higher than the frequency originally received, effectively lowering the energy of the system. The lasers are situated in a way that the Doppler shift works toward the frequency of light being shifted to a frequency slightly below the average emission frequency of the atoms, which would on average lower the energy of the
system. The loss of energy is attributable to a loss in the atoms’ kinetic energy. This results in a lowering of the temperature of the (gaseous) atoms in the trap because temperature is proportional to the average kinetic energy by (4),

\[ .5m < v^2 > = 1.5k_B T. \]  

(This definition of temperature is given because it gives the reader a basic understanding of how energy and temperature are related for ideal gases. A better definition of temperature in regards to Bose-Einstein Condensates is given later in this chapter.) Using laser cooling inside a magneto-optical trap allows temperatures in the micro-Kelvin range to be achieved.

To reach the nano-Kelvin range and BEC’s domain, a process called evaporative cooling is utilized [5]. Because the most energetic bosons can penetrate to the outward regions of the magneto-optical trap, slowly lowering the magnetic field and therefore lowering the exit potential of the trap removes the most energetic bosons from the trap, lowering the overall energy and temperature of the remaining bosons [5] [6] [7]. This is similar to how water evaporating from a cup of coffee lowers the temperature of the coffee and is why the process is coined evaporative cooling. During evaporative cooling approximately 99.9% of the atoms leave the trap because they have too much kinetic energy for BEC formation and are not bosons in the ground state. Figure 2 shows how lowering of the magnetic barrier will lower the overall temperature of the contained atoms and only after the barrier has been lowered enough does BEC form.

![Figure 2](image-url)
Figure 3 shows the density of the BEC as first reported by Eric Cornell and Carl Wieman as it appeared in different stages throughout the experiment.

![Figure 3](image)

**Figure 3 [4]**

### 2.3 Bose-Einstein Condensate Theory

This section of chapter two will provide a theoretic background for Bose-Einstein Condensates by introducing Bose-Einstein Statistics, solving the density of states of bosons in a parabolic trap, and finally, solving for the Bose-Einstein critical temperature when using a parabolic trap. The linear Schrödinger equation is applied.

#### 2.3.1 Bose-Einstein Statistics

A Bose-Einstein distribution is defined by

\[
n(E_i) = \frac{1}{e^{(E_i - \mu)/k_B T} - 1}
\]  

(5)

where \(n(E_i)\) is the average number of bosons in a particular energy state \(E_i\), \(\mu\) is the chemical potential, \(k_B\) is the Boltzmann constant, and \(T\) is the temperature [2] [3]. The system has a specified total number of bosons determined by

\[
\sum_i n(E_i) = \sum_i n_i = N,
\]

(6)

where \(n_i\) is the average number of particles in the ground state (BEC state) [2].
Because Bose-Einstein Condensate is a group of bosons that are in the ground state energy level $E_1$, a value for the chemical potential of BEC is found by evaluating

$$\lim_{T \to 0} n_1 = \lim_{T \to 0} \frac{1}{e^{(E_1 - \mu)/k_B T} - 1} = N \quad (7)$$

and then approximating the exponential and using the first 2 terms of its Taylor Series yields

$$\frac{1}{1 + (E_1 - \mu)/k_B T} - 1 \cong N \quad (8)$$

when $T$ is close to zero [2]. Solving for $\mu$ yields

$$\mu \cong E_1 - \frac{k_B T}{N}. \quad (9)$$

In equation (9), $\mu$ approaches the ground state energy as $T$ approaches zero. Substituting (9) into (7) should give us back approximately $n_1$.

### 2.3.2 Density of States

In this section the density of states is solved for bosons in a parabolic magnetic trap because all BEC experiments are done using a parabolic magnetic trap. The steps taken will follow a similar pattern used to solve for the density of states of an infinite potential box trap in [2]. The density of states function specifies the number of energy configurations that correspond to the energy of a particular energy level, or as seen in

$$D(E) = \frac{dN(E)}{dE}. \quad (10)$$

Different energy configurations correspond to degenerate energy states that are defined by a state that has the same energy but has a differently shaped or rotated wave function. An example of a degenerate energy state is,

$$E_{123} = E_{132}. \quad (11)$$
A 3D parabolic infinite potential well has energy configuration solutions

\[ E_{x_1x_2x_3} = hf (x_1 + x_2 + x_3 + 3/2) \cong hf (x_1 + x_2 + x_3). \]  \hspace{1cm} (12)

Degenerate energy states are important for defining the density of states because some energy levels have several degenerate states allowing electrons to occupy differently shaped and oriented spaces while having the same energy.

The number of possible energy configurations \( e \) for a particular energy level \( L \) for a 3D parabolic potential trap can be determined using

\[ e = \frac{1}{2} (L^2 + L), \]  \hspace{1cm} (13)

where the first energy level is given by \( L = 1 \). Equation (13) is plotted in Figure 4 and shows that if atoms are uniformly distributed over the possible energy configurations for atoms in a parabolic trap, there will more atoms occupying higher energy levels rather than lower energy levels.
To encompass all of the energy configurations that are less than or equal to the energy of a particular energy level $E$ we can look at the graph in Hilbert space of $x_1$, $x_2$, and $x_3$ as seen in Figure 5 where each point inside the tetrahedron represents an energy configuration $E_{x_1 x_2 x_3}$ and $E_{x_1 x_2 x_3} \leq E$. The lowest energy level is $E_{111}$ and is located at the origin of Figure 5. The equilateral triangle in Figure 5 is an equi-energy surface that contains the energy configurations for the energy level $E$ and the number of configurations can be found using equation (13). Equi-energy equilateral triangles can be drawn with vertexes at permutations of an energy state $E_{x_1 x_2 x_3}$. Each point inside the tetrahedron in Figure 5 that is less than or equal to $E$ is written as,

$$x_1 + x_2 + x_3 \leq X = \frac{E}{hf}. \quad (14)$$

The number of energy states less than $E$, is given by finding the volume of the tetrahedron

$$N(E) = \frac{1}{8} \text{Volume Cube} = \frac{1}{8} X^3 = \frac{1}{8} \left( \frac{E}{hf} \right)^3. \quad (15)$$

Plugging (15) into equation (10) yields,

$$D(E) = \frac{1}{8} \frac{d}{dE} \left( \frac{E}{hf} \right)^3 = KE^2. \quad (16)$$
2.3.3 Bose-Einstein Critical Temperature

The Bose-Einstein critical temperature is a temperature at which BEC begins to form. The critical temperature also varies based on the geometry of the magnetic trap because the critical temperature is dependent on the density of states. The formation of BEC can be attributed to atoms moving into the ground state $n_1$. Therefore, to find the critical temperature we will be setting $n_1 = 0$ in equation (17). Following the pattern presented in [2] for a infinite potential box trap, the critical temperature for a parabolic trap is obtained substituting (16) for $D(E)$ as shown in (17)

$$\sum_{i=2}^{\infty} n_i = \int_0^\infty \frac{D(E)}{e^{\frac{E-\mu}{k_BT}} - 1} dE = K \int_0^\infty \frac{E^2}{e^{\frac{E-\mu}{k_BT}} - 1} dE = N. \tag{17}$$

Because $N \approx 10^{20}$ and because $E_1$ is also very small we can approximate

$$\mu = E_1 - \frac{k_BT}{N} \approx 0, \tag{18}$$

for the system. Now $\mu = 0$, the substitution

$$v = \frac{E}{k_BT} \text{ and } dv = \frac{dE}{k_BT} \tag{19}$$

into (17), gives

$$K(k_BT)^3 \int_0^\infty \frac{v^2}{e^v - 1} dv = K(k_BT)^3 2.204 = C T^3. \tag{20}$$

The constant 2.204 is given in the Appendix 6.1. Defining the characteristic temperature $T_b$ for which the integral in (20) is equal to $N$ yields

$$N = C T_b^3 = \frac{(k_BT_b)^3}{8(hf)^3}. \tag{21}$$
By rearranging (21)

$$T_b = \frac{2hfN^3}{k_b}. \hspace{1cm} (22)$$

Because equation (22) and \(n_1 = 0\) when \(T = T_b\), the occupancy of the ground state for a parabolic trap can be expressed as

$$n_1 + \left(\frac{T}{T_b}\right)^3 N = N, \hspace{1cm} (23)$$

and rearranged to obtain

$$\frac{n_1}{N} = \left[1 - \left(\frac{T}{T_b}\right)^3\right]. \hspace{1cm} (24)$$

BEC will not start to form until \(T < T_b\). [2] gives the occupancy of the ground state for a 3D box trap to be

$$\frac{n_1}{N} = \left[1 - \left(\frac{T}{T_{b1}}\right)^{1.5}\right], \hspace{1cm} (25)$$

where

$$T_{b1} = \frac{\hbar^2}{2k\pi m(2.612)^{2/3}} \left(\frac{N}{V}\right)^{2/3} \hspace{1cm} (26)$$

is the critical temperature for a 3D box trap and has dependence on the volume of the box. BEC will not start to form until \(T < T_{b1}\). Equation (24) and (25) are plotted as the red and blue line respectively in Figure 6 and Figure 6 compares \(n_1\)'s relationship to \(\left(\frac{T}{T_b}\right)^3\) and \(\left(\frac{T}{T_{b1}}\right)^{1.5}\) for the parabolic and box traps.
Figure 6 - The occupancy of the ground state as a function of temperature. The red line is for a parabolic trap and the blue line is for a 3D box trap. The MATLAB code is located in Appendix 6.2.
3. “Quantum Phase of a Bose-Einstein Condensate with an Arbitrary Number of Atoms”

This chapter will be devoted to reproducing and diving into the theoretical paper “Quantum Phase of a Bose-Einstein Condensate with an Arbitrary Number of Atoms” by Javanainen and Yoo [1]. Reference [1] considers two incoherent BECs with equal number of atoms that are dropped on top of one another and shows how expected interference atom densities and phase factors between interfering BECs can be built up atom by atom. Their method is novel because it shows how the phase between interfering BECs arrives to a constant phase factor instead of the phase factor just appearing at the end of an interference experiment as seen in equation (31).

In quantum mechanics, there is an uncertainty relation between the number of atoms and the phase of these atoms, therefore as we know less and less about the number of atoms we begin to understand more and more about the phase between these atoms. This uncertainty relation’s final state is still being debated in the physics world but [8] has the Heisenberg limit as

\[
\langle \Delta \phi^2 \rangle \langle \Delta N^2 \rangle \geq 1. \tag{27}
\]

The paper [1] makes use of this uncertainty relation because when detecting photons, the observer does not know which BEC the photon came from so either BEC has \( N/2-1 \) atoms (because absorbing a photon causes that atom to leave the BEC) which increases the uncertainty in the number of atoms but also begins to give us a higher certainty in the interference phase.

3.1 Introduction to Waves and Interference

In 1924 Louis de Broglie reasoned that because light has particle and wave properties that matter would also exhibit particle and wave properties, namely a free particle with rest mass \( m \), moving at speed \( v \), should have a
wavelength \( \lambda \) associated with its momentum [9]. His hypothesis was observed as a diffraction pattern of electrons with known momentum to have a de Broglie wavelength,

\[
\lambda = h \sqrt{\frac{1 - \frac{v^2}{c^2}}{mv}},
\]

(28)

where \( h \) is plank's constant and \( c \) is the speed of light in a vacuum [9]. The fact that matter had wave-like properties gave rise to quantum mechanics and the Schrödinger’s equation because Schrödinger’s equation describes probability waves assigned to the matter in question.

Wave interference is a phenomenon that occurs when coherent waves are added together to form a wave that has altered displacements in the direction of the amplitude based on its constituent parts. Coherent waves are waves that have a constant phase relationship while incoherent waves are waves that have an unknown or random phase relationship. For the two slit experiment, an approximation for the amplitude of the interfering waves at the detection screen is

\[
A = 4A_0 \cos^2 \left( \frac{\pi}{\lambda} (r_1 - r_2) \right),
\]

(29)

where \( r_1 \) and \( r_2 \) are depicted in Figure 7 and represent the the distances from each respective slit to a point along the detection screen [9]. Maximum amplitudes occur where the difference between \( r_1 \) and \( r_2 \) is zero or an integer multiple of the wavelength.

Figure 7: [9]
Essentially, the more the wave crests lay on top of each other, the more their amplitudes add together.

A conventional wave function to describe the interference of two BECs, each having \( N/2 \) atoms with wave numbers set to \( +\pi \) and \( -\pi \), is two plane waves traveling in opposite directions and is,

\[
\psi(x, t) = \frac{N}{\sqrt{2}} e^{-i\omega_k t} (e^{i\pi x + i\phi_+} + e^{-i\pi x + i\phi_-}),
\]

where \( \phi_+ \) and \( \phi_- \) are constant phase factors [1]. Equation (30) has normalization constant and amplitude \( \sqrt{N/2} \), time dependence \( e^{-i\omega_k t} \). Taking the modulus squared of equation (30) gives us an expected interference atom density of

\[
|\psi(x)|^2 = n_0 (1 + \cos(2\pi x + \phi_+ - \phi_-)).
\]

If the two BECs are initially incoherent, the difference in the phase factors will not be known until after the interference fringes have been observed whereas if the two BECs are initially coherent, the difference in the phase factors is calculable before the experiment has taken place (and is discussed in [6]).

### 3.2 Building an Interference Phase Atom by Atom

The Heisenberg picture field operator is another “view” of the Schrödinger picture where instead of wave functions, states, and kets moving and evolving over time, the wave functions are adjusted by applying operators that carry time dependence [3]. “Quantum Phase of a Bose-Einstein Condensate with an Arbitrary Number of Atoms” begins by considering two incoherent BECs with \( N/2 \) spinless, noninteracting bosons residing on a unit interval in one dimension and gives the reader a Heisenberg picture field operator representing plane waves traveling in opposite directions,

\[
\tilde{\psi}(x, t) = e^{i(\pi x + \omega t)} b_\pi + e^{i(\pi x - \omega t)} b_{-\pi},
\]

(32)
but because the time dependence in the expectation value is equal to one, we will ignore it in our calculations giving,

\[ \tilde{\psi}(x_m) = e^{i\pi x_m} b_\pi + e^{-i\pi x_m} b_{-\pi} \] (33)

where \( x_m \) is the position of the \( m \)th detected atom and the wave numbers \( \pm \pi \) are representing the 2 BECs. \( b_{\pm \pi} \) are annihilation operators that act upon the modes \( \pm \pi \) of the number state vector,

\[ |\phi^0\rangle = |N/2_{+\pi}, N/2_{-\pi}\rangle, \] (34)

like,

\[ b_{+\pi} |N/2_{+\pi}, N/2_{-\pi}\rangle = \frac{\sqrt{N}}{\sqrt{2}} |N/2_{+\pi} - 1, N/2_{-\pi}\rangle \] (35)

\[ b_{-\pi} |N/2_{+\pi}, N/2_{-\pi}\rangle = \frac{\sqrt{N}}{\sqrt{2}} |N/2_{+\pi}, N/2_{-\pi} - 1\rangle. \]

It is also useful to note that creation operators act on the number state vector like,

\[ b^\dagger_{+\pi} |N/2_{+\pi}, N/2_{-\pi}\rangle = \sqrt{\frac{N}{2}} + 1 |N/2_{+\pi} + 1, N/2_{-\pi}\rangle \] (36)

\[ b^\dagger_{-\pi} |N/2_{+\pi}, N/2_{-\pi}\rangle = \sqrt{\frac{N}{2}} + 1 |N/2_{+\pi}, N/2_{-\pi} + 1\rangle \]

satisfying the canonical relations \([b_{\pm \pi}, b^\dagger_{\pm \pi}] = 1\).
The number state vector tells you how many bosons are in either BEC at that point in the calculation and there is no uncertainty in the number of bosons in either BEC for the initial state vector. Notice that equation (33) does not take into account a phase factor like the wave function in equation (30) does because the phase will be built up by using the theory of photon detection as a model to find the probable position of the ensuing atom detection.

In [1] Javanainen and Yoo postulate that the joint counting rate for \( m \) atom detections is

\[
R^m(x_1,t_1; \ldots ;x_m,t_m) = K^m \langle \hat{\psi}^\dagger(x_1,t_1) \ldots \hat{\psi}^\dagger(x_m,t_m) \rangle \times \hat{\psi}(x_m,t_m) ... \hat{\psi}(x_1,t_1).
\] (37)

This is a Heisenberg picture expectation value for the product of \( 2m \) boson field operators. \( K^m \) is a constant that depicts the sensitivity of the detectors. The angled bracket notation in (37) implies that (34) is applied to each side of the angled brackets like in this example,

\[
\langle f \rangle = \langle N/2_{+\pi} , N/2_{-\pi} \mid f \mid N/2_{+\pi} , N/2_{-\pi} \rangle.
\] (38)

They find the advantage that \( R^m \equiv 0 \) for \( m > N \) because \( N \) atoms should not trigger more than \( m \) detectors. For theoretical simplicity, they assume that all atoms get recorded at positions \( x_1, \ldots , x_m \) where \( x_1 \) is the first observed boson position and \( x_m \) is the \( m \)th. They also note that when all atoms are recorded the joint counting rate will be proportional to the joint expectation probability,

\[
p^m(x_1, \ldots , x_m) = \frac{(N-m)!}{N!} \langle \hat{\psi}^\dagger(x_1) \ldots \hat{\psi}^\dagger(x_m) \rangle \times \hat{\psi}(x_m) ... \hat{\psi}(x_1).
\] (39)

When the previous positions \( x_1, \ldots , x_{m-1} \) are known and plugged into equation (39), the joint probability becomes a conditional probability for the \( m \)th detection,

\[
p(x_m) = 1 + \beta_m \cos(2\pi x_m + \phi_m),
\] (40)

which has a phase \( \phi_m \) and an amplitude \( \beta_m \) that are dependent on \( x_1, \ldots , x_{m-1} \). Equation (40) has the form of an expected interference atom density like equation (31).
For \( m=1 \) atom detection out of \( N \) atoms,

\[
p^1(x_1) = \frac{(N-1)!}{N!} (\hat{\psi}^\dagger(x_1) \times \hat{\psi}(x_1))
\]  \hspace{1cm} (41)

\[
= \frac{1}{N} \langle N/2_{+\pi} , N/2_{-\pi} | \hat{\psi}^\dagger(x_1) \times \hat{\psi}(x_1) | N/2_{+\pi} , N/2_{-\pi} \rangle
\]

\[
= \frac{1}{N} \langle N/2_{+\pi} , N/2_{-\pi} | (b_{+\pi}^\dagger b_{+\pi} + b_{-\pi}^\dagger b_{-\pi} + b_{+\pi}^\dagger b_{-\pi} e^{(-2i\pi x)} + b_{-\pi}^\dagger b_{+\pi} e^{(2i\pi x)}) | N/2_{+\pi} , N/2_{-\pi} \rangle.
\]

Each term within the parenthesis in equation (directly above) is evaluated individually and then summed and because \( \langle N/2_{+\pi} , N/2_{-\pi} | N/2_{+\pi} , N/2_{-\pi} \rangle \) is synonymous with a scalar product, orthogonality rules apply. The terms are evaluated in equation set (42) and summed in (43):

\[
\langle N/2_{+\pi} , N/2_{-\pi} | (b_{+\pi}^\dagger b_{+\pi}) | N/2_{+\pi} , N/2_{-\pi} \rangle = N/2 \langle N/2_{+\pi} , N/2_{-\pi} | N/2_{+\pi} , N/2_{-\pi} \rangle = N/2,
\] \hspace{1cm} (42)

\[
\langle N/2_{+\pi} , N/2_{-\pi} | (b_{-\pi}^\dagger b_{-\pi}) | N/2_{+\pi} , N/2_{-\pi} \rangle = N/2 \langle N/2_{+\pi} , N/2_{-\pi} | N/2_{+\pi} , N/2_{-\pi} \rangle = N/2,
\]

\[
e^{(-2i\pi x)} \langle N/2_{+\pi} , N/2_{-\pi} | (b_{+\pi}^\dagger b_{-\pi}) | N/2_{+\pi} , N/2_{-\pi} \rangle
\]

\[
= \sqrt{N/2} \left(\frac{N}{2} + 1\right) e^{(-2i\pi x)} \langle N/2_{+\pi} , N/2_{-\pi} | N/2_{+\pi} + 1, N/2_{-\pi} - 1 \rangle = 0,
\]

\[
e^{(2i\pi x)} \langle N/2_{+\pi} , N/2_{-\pi} | (b_{+\pi}^\dagger b_{-\pi}) | N/2_{+\pi} , N/2_{-\pi} \rangle
\]

\[
= \sqrt{N/2} \left(\frac{N}{2} + 1\right) e^{(2i\pi x)} \langle N/2_{+\pi} , N/2_{-\pi} | N/2_{+\pi} - 1, N/2_{-\pi} + 1 \rangle = 0,
\]

and finally we obtain

\[
p^1(x_1) = 1,
\] \hspace{1cm} (43)

which is a uniformly distributed probability density for the position \( x_1 \) of observed boson 1. Setting equation (43) to (40) shows that \( \beta_1 = 0 \) and that \( \phi_1 \) can be any value and is completely uncertain. Boson 1 is no longer a part of either the \(+\pi\) BEC or the \(-\pi\) BEC which allows the current number state vector to be written
\[
\tilde{\psi}(x_1) \mid \psi^0) = \mid \phi^1) = \frac{1}{\sqrt{2}} \left( e^{i\pi x_1} \mid N/2_{+\pi} - 1, N/2_{-\pi} \right) + e^{-i\pi x_1} \mid N/2_{+\pi}, N/2_{-\pi} - 1 \right) \tag{44}
\]

because both possible states are accounted for. Because more states exist as a part of \( \mid \phi^1) \) than \( \mid \phi^0) \), the uncertainty of the state of the BECs is higher in \( \mid \phi^1) \) than \( \mid \phi^0) \), so when \( \mid \phi^1) \) is used to find the joint probability in (45), the certainty in the interference phase of the BECs will increase because (27).

For \( m=2 \) atom detections out of \( N \) atoms,

\[
p^2(x_1, x_2) = \frac{(N - 2)!}{N!} \left( \tilde{\psi}^\dagger(x_1) \times \tilde{\psi}^\dagger(x_2) \times \tilde{\psi}(x_2) \times \tilde{\psi}(x_1) \right) \tag{45}
\]

\[
= \frac{(N - 2)!}{N!} \left( \phi^1 \mid \tilde{\psi}^\dagger(x_2) \times \tilde{\psi}(x_2) \mid \phi^1 \right)
\]

\[
= 1 + \frac{N}{2(N - 1)} \cos(2\pi(x_1 - x_2)),
\]

which is a joint probability because \( x_2 \) has dependence on \( x_1 \). A conditional probability of the form (40) can be found if the value of \( x_1 \) is known and plugged into (45). This will allow us to find a value (which is dependent on \( x_1 \)) for \( \phi_2 \) in the conditional probability for the position \( x_2 \). Because the state vector \( \mid \phi^1) \) contains uncertainty for the number of atoms in either BEC and it is used in equation (45) which then inevitably leads to a conditional probability for the position \( x_2 \), the value for \( \phi_2 \) has some certainty rather than its predecessor, \( \phi_1 \), which has a completely unknown phase because \( \mid \phi^0) \) has full certainty in the number of atoms in either BEC. This explanation follows the uncertainty relationship (27).

After the second atom detection the state vector can be written,

\[
\tilde{\psi}(x_2) \mid \phi^1) = \mid \phi^2) \tag{46}
\]

\[
= \frac{1}{\sqrt{2 + 4 \cos^2(x_1 - x_2)}} \left( e^{i\pi (x_1 + x_2)} \mid N/2_{+\pi} - 2, N/2_{-\pi} \right) + 2\cos(x_1 - x_2) \mid N/2_{+\pi} - 1, N/2_{-\pi} - 1 \right)
\]

\[
+ e^{-i\pi (x_1 + x_2)} \mid N/2_{+\pi}, N/2_{-\pi} - 2) \right).
\]
It is twice as likely that when \( x_1 - x_2 = 0 \) in (46) the BECs will be in a state where both have had one atom removed compared to a state where only one BEC has had two atoms removed, and according to (45) \( x_1 - x_2 = 0 \) is the most likely case. Because more states exist for \( |\phi^2\rangle \) than \( |\phi^1\rangle \) the uncertainty of the state of the BECs is greater for \( |\phi^2\rangle \) than \( |\phi^1\rangle \), so when \( |\phi^2\rangle \) is used to find the joint probability in (47), the certainty in the interference phase of the BECs will increase because of (27). Figure 8 is a plot of \( p^2(x_1, x_2) \) and it illustrates that once the value of \( x_1 \) is known, the value of \( x_2 \) will most likely have a value close to \( x_1 \).

\[
p^3(x_1, x_2, x_3) = \frac{(N - 3)!}{N!} (\hat{\psi}^\dagger(x_1) \times \hat{\psi}^\dagger(x_2) \times \hat{\psi}^\dagger(x_3) \times \hat{\psi}(x_3) \times \hat{\psi}(x_2) \times \hat{\psi}(x_1)).
\]

\[
= \frac{(N - 3)!}{N!} (\phi^2 \mid \hat{\psi}^\dagger(x_3) \times \hat{\psi}(x_3) \mid \phi^2)
\]

Figure 8- MATLAB code is located in Appendix 6.5

For \( m=3 \) atom detections out of \( N \) atoms,
\[ = 1 + \frac{N}{2(N-1)} \left[ \cos(2\pi(x_1 - x_2)) + \cos(2\pi(x_1 - x_3)) + \cos(2\pi(x_2 - x_3)) \right]. \]

Position \( x_3 \) has dependence on \( x_1 \) and \( x_2 \) and equation (47) is a joint probability. If values for \( x_1 \) and \( x_2 \) are known and inserted into (47) it turns into a conditional probability of the form (40) where values \( \beta_3 \) and \( \phi_3 \) depend on the coordinates \( x_1 \) and \( x_2 \).

After three atom detections, and taking \( x_1, x_2, \) and \( x_3 \) to be equal to zero for simplicity, the state vector is

\[ \tilde{\psi}(x_3) \langle \phi^2 \rangle = \langle \phi^3 \rangle \] (48)

\[ = \frac{1}{2\sqrt{3}} \left( \langle N/2+\pi - 3, N/2-\pi \rangle + 3 \langle N/2+\pi - 2, N/2-\pi - 1 \rangle + 3 \langle N/2+\pi - 1, N/2-\pi - 2 \rangle + \langle N/2+\pi, N/2-\pi - 3 \rangle \right). \]

\( \langle \phi^3 \rangle \) is three times more likely to be in a state where both BEC have had at least one atom removed compared to a state where only one BEC has had no atoms removed.

### 3.3 Simulating the Interference of N Atoms

Section 3.2 has discussed the probability distributions for a few atom detections but as the number of detections increases, so does the complexity of the calculation. This section discusses methods used in MATLAB to numerically simulate BEC interference using [1]'s method of building an interference phase. Because \( \beta_m \) and \( \phi_m \) in the conditional probability for the detection \( m \) (40) depend on the previous coordinates, we simulate each coordinate sequentially 1 to \( N \).

Because the position of the first atom being detected has a uniform probability distribution, (43), the first coordinate is randomly generated on the detection interval. After the first position detection, we may calculate the value of \( \beta_m \) and \( \phi_m \) for the \( m \)th probability density by noticing the conditional probability,
\[ p(x_m | x_1, \ldots, x_{m-1}) = \frac{p^m(x_1, \ldots, x_m)}{p^{m-1}(x_1, \ldots, x_{m-1})} = 1 + \beta_m \cos(2\pi x + \varphi_m), \]  

(49)

where \( \beta_m \) and \( \varphi_m \) are functions of \( x_1, \ldots, x_{m-1} \) (but the second detection only depends on \( x_1 \)).

To find the conditional probability in MATLAB we keep track of vector

\[ |v_m\rangle = \left(\frac{(N - m)!}{N!}\right)^{1/2} \hat{\psi}^m |N/2_+, N/2_-\rangle, \]

(50)

which contains the number state vector and if multiplied by its complex conjugate it gives the joint probability,

\[ p^{(m)} = \langle v_m | v_m \rangle. \]

(51)

Equation (50) satisfies,

\[ |v_{m+1}\rangle = (N - m)^{-1/2} \hat{\psi}^m |v_m\rangle, \]

(52)

which gives the joint probability of the \( (m + 1) \)th dependence on the previous detections. To find the conditional probability distribution for the ensuing position of atom \( (m + 1) \), calculate two values \( q_i \) \( (i = 1,2) \) of the joint probability \( p^{m+1} \) for two values \( x_i \) \( (i = 1,2) \) of the coordinate according to the relation

\[ |u_i\rangle = (N - m)^{-1/2} \hat{\psi}(x_i) |v_m\rangle, \]

(53)

\[ q_i = \langle u_i | u_i \rangle \text{ for } i = 1, 2. \]

Find two values, \( p_i \) \( (i = 1,2) \), of the conditional probability \( p \) by the relations,

\[ p_i = \frac{q_i}{\langle v_m | v_m \rangle} \]

(54)

and fit it to the expressions,

\[ p_1 = 1 + \beta \cos(2\pi x_1 + \varphi), \]

(55)
\[ p_2 = 1 + \beta \cos(2\pi x_2 + \varphi) \]

To solve for the ensuing \( \varphi \), transform (55) to the equation,

\[
\left[ (p_1 - 1) \cos(2\pi x_2) - (p_2 - 1) \cos(2\pi x_1) \right] \cos(\varphi) \\
= \left[ (p_1 - 1) \sin(2\pi x_2) - (p_2 - 1) \sin(2\pi x_1) \right] \sin(\varphi),
\]

and because the choice of the two values \( x_i \) (\( i = 1, 2 \)) is arbitrary, in MATLAB chose \( x_1 = 0 \) and \( x_2 = 1/4 \) to simplify the relation (56) to

\[
\tan^{-1} \left[ \frac{p_2 - 1}{p_1 - 1} \right] = \varphi. \tag{57}
\]

Find \( \beta \) with either equation from (55) by plugging in the value of \( \varphi \). Using the acceptance/rejection numerical method in MATLAB, the position of atom \((m + 1)\) is found. The acceptance/rejection method is used for randomly generating distributions that match uncommon probability distribution functions like \( 1 + \beta \cos(2\pi x_1 + \varphi) \). The acceptance/rejection method randomly generated two values that are either accepted or rejected depending on if they fit the interference probability distribution. The MATLAB in its entirety can be found in Appendix 6.5.

### 3.4 Interpretation of Results

Figure 9 is a histogram that depicts atom density from a simulation that had 50 atom position detections.
and the solid line is a best fit interference curve of the form (40) or (31). The x-axis is broken up into 30 bins, each with width 1/30th of the axis. The y-axis is the number of atoms that have been located in each bin. The number of atoms accumulates in a bin when the position of a detected atom is inside the range of that bin. Figure 9 shows that it is difficult to make out a coherent interference pattern. This is due to a high degree of fluctuation in the simulated value of $\varphi_m$ for that detection relating to the conditional probability (40) for that detection. Figure 10 shows $\varphi_m$ for each $m$ and that there is large fluctuation in the value of $\varphi_m$ for initial detections.

![Figure 10](image)

Because $\varphi$ fluctuates from detection to detection, the conditional probability density (40) is shifting left to right making atom positions more randomly distributed in relation the best fit curve on Figure 9. It is evident that $\langle \varphi \rangle$ still has a high uncertainty value during initial atom detections because the value of $\varphi$ distinctly changes from detection to detection which follows from (27) and the fact that $\langle N \rangle$ increases with more atom detections. Figure 11 shows that the value of $\beta_m$ for each $m$ and that $\beta_m$ increases to 1 after a number of detections. As $\beta_m$ increases to 1, the conditional probability density (40) for the next atom detection becomes less like a flat uniform probability density and more dependent on coordinate, appearing as an interference expectation atom density (31). Because a reasonable percentage of the 50 detections in Figure 9 are correlated with a $\beta_m < \sim .9$, the histogram does not
match a best fit curve well.

Figure 12 is a histogram for the simulation of 500 atom position detections and was chosen to be contrasted with Figure 9 because the histogram in Figure 10 better matches a best fit curve than does the histogram in Figure 9.
By the 100th detection, the value of $\varphi$ begins to approach a constant number (can be seen in Figure 13), which exerts a succession of similar conditional probability distributions for successive atom positions, which in turn, creates less fluctuation between the histogram and the best fit curve.

Figure 13 shows that $\beta_m$ has been essentially equal to 1 for many number of detections and because of (40) the position of the $m$th atom has been completely dependent on the coordinates of previous atom positions, making Figure 12’s histogram fit a best fit curve better than Figure 9’s.
Figure 15 is a histogram for the simulation of 2000 atom position detections (the most my computer could reasonably generate) and it matches its best fit curve very well because $\beta_m$ has been approximately equal to 1 since the $m = 60^{th}$ detection and $\varphi_m$ has been essentially constant from detection to detection for many detections. Figures 16 and 17 show the value of $\beta_m$ and $\varphi_m$ respectively for each $m$.
Figures 9, 12, and 15 show the trend that histograms will better match their respective best fit interference atom density curve as the number of atom detections increase. This trend can be extended to simulations with more than 2000 atom detections.
4. Conclusions

After an atom detection on two overlapping incoherent BEC subsystems, we know nothing about the actual number state of each subsystem but have maximal knowledge of the whole because either BEC can have lost 1 atom as seen in (44). This implies that the two BECs are in a Bell state and that they are entangled [10].

Entanglement occurs when two or more systems interact. As more bosons are detected in interfering BECs, the two BECs become more entangled because the number of pure number state vectors needed to describe the whole system grows. We can conclude that it is the detection laser that is entangling the overlapping BECs. We can also conclude that the interfering bosons are being “shared” by both BEC because they do not belong to either BEC but interact and are entangled with both of them. Once all of the atoms have been detected, all of the atoms are entangled and we are certain that no atoms belong to either BEC but are completely uncertain which number state “path” it took to get there.

We know that BECs exist below the critical BEC temperature and that when we are observing BECs (via photons), the agitated boson leaves the BEC (because it is excited out of the ground state). Therefore we can conclude that if we were to simultaneously shine a laser on all particles in the BEC, the former BEC would have a temperature greater that the critical BEC temperature, instantaneously as they have absorbed energy from the photons and are no longer in the ground state.

Javanainen and Yoo’s paper suggests that BECs behave as if it had a phase as soon as there is a large enough occupation number in an individual quantum state and that no interactions between the atoms in the BEC are needed to communicate the phase of the BEC. If one envisions BEC as an ensemble of separate single identical boson systems under the same shape potential in the ground state that are all superimposed on top of one another (because bosons can occupy the same state) to form the BEC’s wave function, the previous statement loses some mystery because the phase of the BEC and ground state would be determined by the potential and its atoms just happening to statistically “fall into place” due to the potential barrier and the nature of a
probability wave function (instead of a phase being communicated). The more atoms in the condensate will make the phase and wave function more vivid, just like an ensemble of particles is needed to fill all the gaps of a wave function for it to be expressed in experiment.
5. References


6. Appendixes:

This chapter is referenced throughout the document so the document may retain flow.

6.1 Substitution for Equation (21)

\[
\int_0^\infty \frac{x^{p-1}}{e^x - 1} \, dx = \int_0^\infty e^{-x} (1 - e^{-x})^{-1} x^{p-1} \, dx = \int_0^\infty e^{-x} \left[ \sum_{m=0}^{\infty} (e^{-x})^m \right] x^{p-1} \, dx
\]

\[= \left( \sum_{n=1}^{\infty} \frac{1}{n^p} \right) \times \int_0^\infty e^{-y} y^{p-1} \, dy = \xi(p) \times \Gamma(p) \] [2]

With substitutions \(n=m+1\) and \(y=nx\). \(\xi(p)\) is the Riemann zeta function and has known values for some \(p\) values and \(\Gamma(p)\) can be solved for numerically [2]. In equation (21) we use \(p=3\) and obtain

\[\xi(3) \times \Gamma(3) = 2.204\]

<table>
<thead>
<tr>
<th>(p)</th>
<th>(\xi(p))</th>
<th>(\Gamma(p))</th>
<th>(\xi(p) \times \Gamma(p))</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.202</td>
<td>2</td>
<td>2.204</td>
</tr>
</tbody>
</table>

The values in the table above are found in [2].

6.2 Matlab Code for Figure 6

```matlab
T = 0:.01:1;
R=T.*T.*T
N=(1-(R));
plot(T,N,'-r')
xlabel('T/T_b and T/T_b_1')
ylabel('Expected (n_1)/(N)')
hold on
W= 0:.01:1;
E=W.^(1.5);
K=(1-E);
plot(W,K, '-b')
```

6.3 MATLAB for \(P(x_1,x_2)\)

```matlab
u=0:.01:1
[X1,X2]=meshgrid(u,u);
N=1000;
Z=(1/999000).*(N^2-N+.5.*N^2).*cos(2*pi*(X1-X2));
surf(X1,X2,Z)
shading interp
colormap cool
xlabel('x1')
ylabel('x2')
zlabel('Probability Density')
```
6.4 MATLAB for Interference of BECs

function [X,beta,phi] = fringes(N)
% calculate set of detection points x_1, x_2, ... x_N for
% N atoms in 2 BEC clouds (N/2 in each) following Javanainen's PRL 1996 paper.

% INPUT:
% N - number of atoms in 2 BEC clouds (even integer)
% OUTPUT
% X - length N vector of detection points.
% beta, phi (length N vectors) - parameters of conditional probability
% distribution for different numbers of detected atoms.
% INITIAL SETUP
% N must be even

if mod(N,2)> 0
    error('N should be even')
end
% allocate arrays of detection coordinates X, joint probabilities jprob,
% betas and phis
X = zeros(N,1);
jprob = zeros(size(X));
beta = zeros(size(X));
phi = zeros(size(X));

% initial state vector of the system, an (N/2+1) x (N/2+1) matrix with
% v_{in}(i,j) = A_{n_{+} = i-1,n_{-} = j-1} being amplitudes of the
% number states |n_{+},n_{-}> (0 <= i,j, <= N/2).
v_{in} = zeros(N/2+1,N/2+1); v_{in}(N/2+1,N/2+1) = 1;

% START THE RUN
m_already_detected = 0;
% generate the first detection coordinate
beta(1) = 0; phi(1) = 0;
X(1) = detect_atom(beta(1),phi(1));
% calculate state vector after the first detection
v_{in} = apply_field_operator(v_{in},X(1),m_already_detected);
% find probability of detecting 1 atom at coordinate X(1)
jprob(1) = v_{in,:}'*v_{in,:}; % we know it must be one

results(30) = 0;
binVals(31) = 0;
binVals(1) = 0;
for i=2:31;
    binVals(i) = binVals(i-1)+ 1;
end
for i=1:30
    if (binVals(i) <= 30*X(1) & 30*X(1) < binVals(i+1));
        results(i) = results(i) + 1;
        found = i;
        break
    end
end
% continue
for m = 2:N
    [beta(m), phi(m)] = find_cond_prob(v_in, jprob(m-1), m - 1);
    % generate new detection coordinate
    X(m) = detect_atom(beta(m), phi(m));
    % calculate new state vector after the detection of the m-th atom
    v_in = apply_field_operator(v_in, X(m), m-1);
    % find new joint probability
    jprob(m) = v_in(:)'*v_in(:);
end

for i=2:31;
    binVals(i) = binVals(i-1) + 1;
end
for i=1:30
    if (binVals(i) <= 30*X(m) & 30*X(m) < binVals(i+1));
        results(i) = results(i) + 1;
        found = i;
        break
    end
end
end

plot(results,'ok')

function [beta, phi] = find_cond_prob(v_in, previous_joint_prob, m_already_detected)
% calculate parameters beta, phi of conditional probability
% \( p(x) = 1 + \beta \cos(2\pi x + \phi) \)
% \( x_s(1) = 0 \)
% \( v_{\text{tmp}} = \text{apply\_field\_operator}(v_{\text{in}}, x_s(1), m_{\text{already\_detected}}) \)
% \( p_s(1) = v_{\text{tmp}}(:,)'*v_{\text{tmp}}(:,); \)
% \( x_s(2) = 1/4 \)
% \( v_{\text{tmp}} = \text{apply\_field\_operator}(v_{\text{in}}, x_s(2), m_{\text{already\_detected}}) \)
% \( p_s(2) = v_{\text{tmp}}(:,)'*v_{\text{tmp}}(:,); \)
% \( \text{cond\_probs} = p_s/\text{previous\_joint\_prob}; \)
% \( [\beta, \phi] = \text{find\_beta\_phi}(x_s, \text{cond\_probs}); \)

function [beta, phi] = find_beta_phi(xs, ps)
% finds parameters of the distribution \( p(x) = 1 + \beta \cos(2\pi x + \phi) \)
% given two values of x and corresponding values of p(x)
% Find phi from the equation \( A\cos(\phi) = B\sin(\phi) \)
A = (ps(1) - 1)*cos(2*pi*xs(2)) - (ps(2) - 1)*cos(2*pi*xs(1));
B = (ps(1) - 1)*sin(2*pi*xs(2)) - (ps(2) - 1)*sin(2*pi*xs(1));
phi = atan2(A, B);
beta = (ps(1) - 1)/cos(2*pi*xs(1) + phi);

function v_out = apply_field_operator(v_in, x, m_already_detected)
% v_out = apply_field_operator(v_in, x, m_already_detected)
% Applies field operator at the coordinate x to the state vector v_in
% of a system with m atoms already detected
% INPUT:
% v_in - input state vector of the system (n x n matrix)
% m_already_detected - number of atoms already detected
% OUTPUT:
% v_out - output state vector of the system (n x n matrix)

% Vector v_in is a square matrix with the entries (0 <= i,j <= N/2)
% v_in{i,j} = A_{n_{+} = i-1,n_{-} = j-1} being amplitudes of the
% number states |n_{+},n_{-}>.

n = size(v_in,1); \% n = N/2 + 1

% form (n x n) matrix for the annihilation operator b_{+}
sqrt_line = sqrt(1:n)'; \% column vector [sqrt(1),sqrt(2),... sqrt(n)]^T;
b = repmat(sqrt_line,1,n);
% circularly shift v_in and zero the row with n_{+} = n
tmp = circshift(v_in,[-1,0]);
tmp(end,:) = 0;
% introduce phase multiplicand
phase_factor = exp(1i*pi*x);
% find result of action on v_in by operator exp(i*pi*x)*b_{+}
v_out = phase_factor*tmp.*b;

% find result of action on v_in by operator exp(-i*pi*x)*b_{-}
tmp = circshift(v_in,[0,-1]);
tmp(:,end) = 0;
v_out = v_out + conj(phase_factor)*tmp.*b';

% dividing by (N-m)^{1/2}
v_out = (2*n - 2 - m_already_detected)^(-1/2)*v_out;

---

function x = detect_atom(beta,phi)
% generates random number x in the range 0 <= x <= 1 with the probability
% p(x) = 1 + beta*cos(2*pi*x + phi) using rejection method (see "Numerical Recipes").
% Comparison function f(x) = 1 + |beta|.

A = 1 + abs(beta);
while 1
    x = rand(1);
    y = A*rand(1);
    if y < 1 + beta*cos(2*pi*x + phi)
        break
    end
end

6.5 Quantum Phase of a Bose-Einstein Condensate with an Arbitrary Number of Atoms
Quantum Phase of a Bose-Einstein Condensate with an Arbitrary Number of Atoms

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We study the interference of two Bose-Einstein condensates within an elementary model. The detection of the atoms is modeled by adapting the standard theory of photon detection. Even though the condensates are taken to be in number states with no phases whatsoever, our stochastic simulations of atom detection produce interference patterns as would also be predicted on the basis of the phases of the macroscopic wave functions describing the condensates. In statistical mechanics terms, we have devised a method to analyze spontaneous symmetry breaking for an arbitrary (not necessarily larger) number of particles.

PACS numbers: 03.75.Fi, 05.30.–d, 32.80.Pj

The confluence of laser cooling and evaporative cooling [1] has recently lead to the first observations [2] of a weakly interacting Bose-Einstein (BE) condensate. Some of the current theoretical work on the optical properties of the condensate [3] and on the consequences of the interparticle interactions [4,5] will undoubtedly soon be tested experimentally. The analogy to lasers [6] should also guarantee that the phase, coherence, and potential for interference of a BE condensate will attract much attention.

In fact, it is customary to attribute to the condensate a macroscopic wave function [5,7] with a magnitude and phase. Essentially, the same approach lends itself to elementary textbook discussions of the Josephson effect [8]. Recognizing this connection, we some time ago predicted oscillatory exchange of atoms between two trapped BE condensates that depends on the phases of the macroscopic wave functions [9]. More recently, we have discovered that no phase is needed at all: The atoms will oscillate even if the condensates are initially in number states, provided the atom numbers are “large enough” [10]. In this Letter we take the next, final, conceptual step. We study the interference of atoms that results when two BE condensates are dropped on top of each other. The example is different from that of Refs. [9] and [10], because in the present case we may adapt a plausible quantum measurement theory for the positions of the atoms from the well-established theory of photon detection. We simulate stochastically the outcome of an experiment. We find that the atoms display an interference pattern as would be deduced from the phases of the wave functions of the condensates, even though no phases have ever been assumed. In effect, we are now able to discuss the consequences of spontaneously broken phase symmetry for an arbitrary atom number.

We take $N$ spinless, noninteracting bosons residing on a unit interval in one dimension. The Heisenberg picture field operator is

$$\hat{\psi}(s,t) = \sum_k e^{i(kx-\omega_k t)} b_k,$$

where the sum runs over wave numbers, $b_k$ is the annihilation operator for the mode $k$, and $\omega_k$ is the mode frequency. The $N$ atoms are divided into two condensates, $N/2$ atoms each. We assume that the condensates have been given pushes in opposite directions, so that the one-particle states $\pm \kappa$ have $N/2$ atoms in them. Other one-particle states are empty. We thus write the state vector as

$$|\phi_0\rangle = |(N/2)_{+\kappa},(N/2)_{-\kappa}\rangle.$$

To simplify the notation further, we arbitrarily set $\kappa = \pi$. Then all of our results are periodic in position with the period of 1. We also take the characteristic frequencies $\omega_{\pm\kappa}$...
to be the same, which will remove all time dependence from the results.

We now need a quantum measurement theory for the positions of the atoms. The well-known theory for photon detection [11] furnishes us with a model. In the standard version it is assumed that each photon is absorbed (removed) upon detection, and that the matrix element for photon absorption is independent of photon energy. The theory then produces the joint counting rate at times $t_1, \ldots, t_m$ for photon counters positioned at $r_1, \ldots, r_m$ as an $m$-time correlation function of the electric field operator. Mutatis mutandis, we posit that in our case, under the same assumptions, the joint counting rate for $m$ atom detectors is

$$R^m(x_1, t_1; \ldots; x_m, t_m) = K^m \langle \hat{\Psi} \dagger (x_1, t_1) \cdots \hat{\Psi} \dagger (x_m, t_m) \hat{\Psi} (x_1, t_1) \cdots \hat{\Psi} (x_m, t_m) \rangle,$$  

(3)

Here we define $[m/2] = m/2$ for even $m$ and $[m/2] = (m-1)/2$ for odd $m$. The functions $C^m_q$ are

$$C^m_q(x_1, \ldots, x_m) = \sum \cos[2\pi(x_{a_1} + \cdots + x_{a_q} - \cdots - x_{a_q})],$$  

(5)

where the sum runs over all sets of distinct indices $\{a_1, \ldots, a_{2q}\}$ chosen from the set $\{1, \ldots, m\}$, but taking only one permutation of each $q$-tuple $\{a_1, \ldots, a_q\}$ and $\{a_{q+1}, \ldots, a_{2q}\}$; we set $C^0_q = 1$.

By construction, the joint probabilities are non-negative and normalized. An explicit calculation shows that they are also compatible:

$$\int p^m(x_1, \ldots, x_{m-1}, x_m) dx_m = p^{m-1}(x_1, \ldots, x_{m-1}).$$  

(6)

This condition, which is usually not discussed in the theory of photon detection, is crucial in order that the conventional theory of probability may be relied on. Finally, let us consider the probability $p^m$ as a function of a particular individual variable $x = x_i$ with the other variables held fixed. It is obvious from Eqs. (4b) and (5) that $p^m$ is a linear combination of a constant, $\cos(2\pi x)$, and $\sin(2\pi x)$. Because the probabilities are non-negative, $p^m(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_m)$ must thus be a constant multiple of a function of the form

$$p(x) = 1 + \beta \cos(2\pi x + \varphi).$$  

(7)

In this case $\beta$ and $\varphi$ are parameters that depend on the fixed coordinates $x_j$ with $j \neq i$.

Our plan is to simulate an experiment by generating an $N$-tuple of random numbers $x_1, \ldots, x_N$ with the probability distribution $p^N(x_1, \ldots, x_N)$. In general, production of random deviates with a prescribed probability density in $N$-dimensional space rapidly becomes a hopeless proposition as $N$ increases. The present task, though, is facilitated by the observation that the conditional probability density for $x_m$ with $x_1, \ldots, x_{m-1}$ fixed, $p(x_m|x_1, \ldots, x_{m-1}) = p^m(x_1, \ldots, x_m)/p^{m-1}(x_1, \ldots, x_{m-1})$, is also of the form (7). First, we have $p^1(x) = 1$, so we obtain $x_1$ as a uniformly distributed random number in the interval [0, 1]. Next, having already generated $m - 1$ coordinates $x_1, \ldots, x_{m-1}$, we simply calculate $p(x|x_1, \ldots, x_{m-1})$ for two different $x$, determine the parameters $\beta$ and $\varphi$ of the function $p(x)$ in Eq. (7) from the results, and use the ensuing $p(x)$ as the distribution from which to draw the subsequent position $x_m$. As a technical detail, it is probably unwise to use the combinatoric formulas (4b) and (5) for numerical purposes. Instead, we obtain the probabilities $p^m$ directly as quantum expectation values, as in Eq. (4a). All told, we have an $N^3$ algorithm for generating $x_1, \ldots, x_N$.

An example is given in Fig. 1(a) for $N = 1000$ atoms. We sort the positions $x_1, \ldots, x_N$ into $n_b = 30$ bins of equal width $\Delta x = 1/n_b$, and plot the histogram of the numbers of atoms falling in each bin using the centers of the bins as the abscissas. We also plot as a continuous line the histogram derived from the probability distribution (7) that gives the best least-squares fit to the simulation histogram, with $\beta$ and $\varphi$ treated as the free
parameters. Both histograms in effect depict one period of a cosine wave with a nearly 100% modulation depth.

Remarkably, even though the probability density for detecting an individual atom \( p^m(x) = 1 \) has no structure at all, an experiment that records all \( N \) atoms at once would nonetheless find an interference pattern with bands of higher and lower atom density. This is a manifestation of the correlations between atomic positions embodied in the probabilities \( p^m \). In our example the atom density is essentially of the form \( n(x) = n_0[1 + \cos(2\pi x + \varphi)] \). If the experiment were repeated, the result would qualitatively be the same; the phase \( \varphi \) just varies at random from one run to the next.

We now contrast our simulations with the conventional reasoning about the phase of a BE condensate. One would ordinarily grant each condensate a macroscopic wave function, and write the total wave function of the two condensates as

\[
\psi(x, t) = \sqrt{\frac{N}{2}} \left( e^{-i\omega t}(e^i\pi x + i\phi_+ + e^{-i\pi x + i\phi_-}) \right).
\]

The phases \( \phi_{\pm} \) are due to spontaneous breaking of phase or “gauge” symmetry [7]. They are independent, fixed for each experiment, but vary randomly from one experiment to the other. In a single experiment with fixed phases \( \phi_{\pm} \), so goes the argument, one expects an atom density of the form \( n(x) = n_0[1 + \cos(2\pi x + \phi_+ - \phi_-)] \); i.e., an interference pattern.

This naive model may be put more rigorously. For instance, one may formally replace the quantum fields describing the condensates by classical fields with the random phases \( \phi_{\pm} \). Alternatively, one may retain the quantum fields, but postulate that the condensates are in the coherent states \( |\alpha_{\pm}\rangle \) with \( \alpha_{\pm} = \sqrt{N/2}e^{i\phi_{\pm}} \) instead of the number states. Whichever way one elects to proceed, conventional arguments lead to the prediction that, as a result of spontaneously broken phase symmetry, the two condensates combine to give an interference pattern with the density \( n(x) = n_0[1 + \cos(2\pi x + \phi_+ - \phi_-)] \). We have illustrated this in Fig. 1(b) by plotting the same histograms as in Fig. 1(a) for \( N = 1000 \) atoms drawn independently from the probability distribution \( p(x) = 1 + \cos(2\pi x + \phi_+ - \phi_-) \) for certain fixed values of \( \phi_+ \).

Our measurement theory and the conventional arguments give very similar atom densities [see Figs. 1(a) and 1(b)]. However, there is a crucial conceptual difference. In any derivation based on spontaneous symmetry breaking, the quantity corresponding to the broken symmetry is ultimately inserted by hand into the analysis. The phases \( \phi_{\pm} \) are a representative example. On the other hand, the phase \( \varphi \) analogous to \( \phi_+ - \phi_- \) emerges as a result from our approach. In this sense we have predicted spontaneous symmetry breaking.

Admittedly it is possible to “predict” spontaneous symmetry breaking by assuming the presence of a symmetry breaking field, then going to the thermodynamic limit, and finally letting the symmetry breaking field vanish [7]. A quantity corresponding to the broken symmetry survives this particular sequence of limits without vanishing. However, for a BE condensate the symmetry breaking field is a mathematical fiction and does not correspond to any physical quantity at all. Our earlier approach [10] did away with the symmetry breaking field, but was still based on the limit of large particle number. The novelty of the present work lies in the fact that, by adopting an explicit measurement theory for the positions of the atoms, we have freed our argument from any semblance of the thermodynamic limit as well.

The question to what extent our measurement theoretical predictions and the broken-symmetry predictions can be distinguished in detail elsewhere [12]. Here we offer only a few qualitative remarks. For \( N = 1000 \) there is no obvious difference between Figs. 1(a) and 1(b). When the number of atoms decreases, the quality of histograms such as those in Fig. 1 deteriorates, and it becomes hard to pick up any interference pattern in the first place. All told, for small \( N \) one must fall back on statistical analysis of repeated experiments. The number of repetitions needed to gather enough statistics to distinguish between the two theories increases rapidly with \( N \), and may be expected to be in the thousands for \( N \) as small as a few tens.

Our results suggest an intriguing angle to the evolution of the phase of the wave function of a BE condensate: The condensate behaves as if it had a phase as soon as there is a large occupation number of an
individual quantum state. No interactions between the atoms are needed to communicate the phase throughout the condensate. Evaporative cooling depends on elastic collisions between the atoms, so this point may seem moot. However, we emphasize that the phase would appear instantaneously even for completely noninteracting atoms if they could be put to the same quantum state with, say, laser cooling. Our views about the role of the interactions are somewhat different from those underlying the ongoing work on the dynamics of BE condensation (see Ref. [13], and references therein).

Our quantum model is clearly simplistic. In recent experiments [2] the condensate was confined to fairly small dimensions, ~1–10 μm. The condensate is modeled more accurately by a large occupation number of the ground state of an atom trap than of a momentum eigenstate. When released in free space, such a condensate flies apart ballistically. Interference effects are lost on a time scale for which we do not yet have an estimate. Besides, interactions between the atoms, weak as they are, may strongly affect the properties of the condensate [4,5]. Apart from these complications, our thought experiment could, perhaps, be realized by launching two condensates with small momenta toward one another, and letting the combining atom clouds fall on an array of position detectors. Interference is essentially one dimensional, taking place in the direction of the momentum difference between the clouds. Our assumption of one spatial dimension thus has some physical validity, and it could be avoided straightforwardly if a need arises. Finally, the units of length and wave number in our presentation are trivial (and actually somewhat contradictory) conventions. This could be corrected easily, at the expense of some additional notation.

We envisage our ideas leading to general practical tools for the analysis of phase and interference phenomena in BE condensates and atom lasers. For instance, the effects of the finite size of the condensate and of the interactions between the atoms could be studied. A calculation of the entire detection statistics for such situations admittedly seems to be a tall order, but we anticipate that already the lowest correlation functions $p^1$ and $p^2$ might give a quantitative estimate of the potential for interference.

In summary, we have presented a new method for the analysis of the interference phenomena associated with a Bose-Einstein condensate. The idea is to compute the joint probability distribution of atom detection for all the atoms at once, and then generate random samples from this distribution for inspection. We have demonstrated that we may predict an interference pattern conventionally attributed to the phase of the condensate without ever assuming a phase. We envisage applications of our ideas to the study of the contrast of the interference, or of the “condensate fraction,” also in more complicated situations involving spatial profiles and atom interactions in a condensate. Finally, couched in statistical mechanics language, we have devised a method to investigate spontaneous symmetry breaking for a finite number of particles. There is no need to go to the thermodynamic limit.

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