A computational basis for approximating the conductivity in electrical impedance tomography

A Major Qualifying Project Report:

Submitted to the Faculty

of the

WORCESTER POLYTECHNIC INSTITUTE

In partial fulfillment of the requirements for the

Degree of Bachelor of Science

By

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Acknowledgements

I would like to thank my advisor, Professor Sarkis for his patience, advice, and guidance during the project. I would also like to thank Professor Tarek Mathew for several of the ideas introduced in this project.
Abstract

This project is concerned with the mathematical reconstruction problem of identifying the unknown conductivity (diffusion) coefficient in an elliptic equation, given full or partial measurements of the Dirichlet to Neumann map on the boundary. This is a nonlinear problem. The potential is calculated by extending the boundary data into the domain. We consider both harmonic and non-harmonic extensions. A basis using sinusoids in the angular direction and polynomials in the radial direction is used to express the extensions. The inner products of the gradients of the harmonic basis functions then act as the basis set for expanding the conductivity. The numerical tests of convergence in the expansion is studied as the number of basis functions is increased, for the problem on the unit disk in two dimensions.
Executive Summary

This project is concerned with the mathematical reconstruction problem of identifying the unknown conductivity (diffusion) coefficient in an elliptic equation, given full or partial measurements of the Dirichlet to Neumann map on the boundary. This is a nonlinear problem. The potential is calculated by extending the boundary data into the domain. We consider both harmonic and non-harmonic extensions. A basis using sinusoids in the angular direction and polynomials in the radial direction is used to express the extensions. The inner products of the gradients of the harmonic basis functions then act as the basis set for expanding the conductivity. The numerical tests of convergence in the expansion is studied as the number of basis functions is increased, for the problem on the unit disk in two dimensions.
CHAPTER 1: Introduction

This project is concerned with the mathematical reconstruction problem of identifying the unknown conductivity (diffusion) coefficient in an elliptic equation, given full or partial measurements of the Dirichlet to Neumann map on the boundary. This is a nonlinear problem. The potential is calculated by extending the boundary data into the domain. We consider both harmonic and non-harmonic extensions. A basis using sinusoids in the angular direction and polynomials in the radial direction is used to for expressing the extensions. The inner products of the gradients of the harmonic basis functions then act as the basis set for expanding the conductivity. The numerical tests of convergence in the expansion is studied as the number of basis functions is increased, for the problem on the unit disk in two dimensions.

Since the coefficient can be treated as conductivity, the reconstruction problem is relevant to electrical impedance tomography (EIT). The partial differential equations used in modelling the conducting object are detailed by L. Borcea. In this project, we work with a simplified form of the complete system of equations she described. We restrict the problem to the unit disk and use continuous boundary data. This is equivalent to the case in which there are no gaps between electrodes.

The PDE problem that we are numerically solving here is:

\[
\begin{cases}
-\nabla \cdot (\sigma \nabla u) = 0 \text{ on } \Omega \\
u = u_\Gamma \text{ on } \partial \Omega , \text{ where } u \in H^1(\Omega).
\end{cases}
\]

Our main focus is on how to obtain \( \sigma \) given the map \( u_\Gamma: \to \sigma \partial_\mathbf{n}u|_\Gamma \). To estimate it, we have to first obtain the discretized Dirichlet to Neumann map, \( S \), through a weak formulation. It maps boundary voltages to currents. \( Su_\Gamma = \sigma \frac{\partial u}{\partial \mathbf{n}} \). We obtain a weak form by multiplying the first equation with a test function, \( v \in H^1(\Omega) \), and applying Green’s theorem:

\[
0 = -\int_\Omega \nabla \cdot (\sigma \nabla u) v dx = \int_\Omega \sigma \nabla u \cdot \nabla v dx - \int_{\partial \Omega} \hat{\mathbf{n}} \cdot (\sigma \nabla u) v dx,
\]

\( \hat{\mathbf{n}} \) is a unit vector normal to \( \partial \Omega \).
We now consider a discretization for the map, $S$. For the unit disk domain ($\Omega = D(0,1)$), we apply Galerkin discretization and search for $u_h \in V_h \subset H^1(\Omega)$. $V_h$ is a finite dimensional subspace of $H^1(\Omega)$. The subscript, $h$, captures how fine a partitioning of $H^1(\Omega)$, when making $V_h$, is. Because $0 = \int_\Omega \sigma_h \nabla u_h \cdot \nabla v_h \, dx - \int_{\partial\Omega} (\sigma_h \nabla u_h \cdot \hat{n}) v_h \, dx$, the Dirichlet to Neumann map is

$$(S_h u_h, v_h) = \int_\Omega \sigma_h \nabla u_h \cdot \nabla v_h \, dx.$$  

Here, $\sigma_h$ is the is an approximation of $\sigma$ so that $u_h$ can be computed inside $\Omega$.

We hope to use a discrete version, $S_h$, to estimate $\sigma_h$ and to obtain extensions of boundary data into the interior domain. As we shall see, this map is not generally invertible in terms of finding $u_h$ or $\sigma_h$. In addition it is nonlinear in both $u_h$ and $\sigma_h$. So, we follow an iterative scheme in which we make an initial guess that $\sigma_h^{(1)} = 1$. This gives a harmonic solution, $u_h$, as the extension of $u_h$ given on $\partial\Omega$. With this harmonic extension, we calculate a new conductivity, $\sigma_h^{(1)}$. This will be an improved guess, in which information from the boundary condition is incorporated.

From the above definition of $S_h$, we can construct a matrix that gives the trace of $u_h$ on the boundary. In order to extend $u_f$, we are faced with the problem that this matrix used to obtain the trace is not necessarily invertible. When constructing $u_h$, we employ a method of minimization with Lagrange multipliers. We make the resulting matrix injective, by only admitting the set of functions in $H^1(\Omega)$ with average zero on $\partial\Omega$, i.e. by excluding the constant function from this set.

Having $u_h$, we can then estimate $\sigma_h$. When approximating $\sigma_h$, we apply a perturbation to linearize to the partial differential equation because the it is nonlinear. The linearization was described by Calderon$^4$. This gives us an iterative method because the linearization entails a hierarchical expansion of the conductivity. So, we have:

$$
\begin{align*}
\left\{\begin{array}{l}
(S_h^{(1)} u_h, v_h) = \int_\Omega \sigma_h^{(1)} \nabla u_h \cdot \nabla v_h \, dx, \\
(S_h^{(2)} u_h, v_h) = (S_h^{(1)} u_h, v_h) + \int_\Omega \tilde{\sigma}_h \nabla u_h \cdot \nabla v_h \, dx, \quad \tilde{\sigma}_h = 0 \text{ on } \partial\Omega \\
(S_h u_h, v_h) = \int_\Omega \sigma_h \nabla u_h \cdot \nabla v_h \, dx \approx \int_\Omega \sigma_R \nabla u_h \cdot \nabla v_h \, dx = (S_h^{(2)} u_h, v_h)
\end{array}\right. 
\end{align*}
$$
The recovered conductivity is $\sigma_R$. Note that, from the above equations, we effectively fix the value of $\sigma_R$ on the boundary since the voltages and currents on the electrodes are assumed to be known.

To solve for $\sigma_R$, we will look at two methods. One way is to determine the true size of the space for $u_\Gamma$ and appropriately match this to the finite element space we search for $\sigma_R$ in order to obtain an injective map for $S_h^{(2)}$. With this method, one adequately limits the number of eigenfunctions that participate in the expansion of $\sigma_R$. Here, there is more freedom in choosing the number of eigenfunctions that participate in the expansion of $\sigma_R$. The other way is to arbitrarily specify the basis functions to use. Both methods solve a least squares minimization problem.

We check for the validity of our method by employing known functions whose values are known on the boundary. These functions are expressed using our chosen basis set of eigenfunctions. Also the conductivity is specified and we try to see how well the first method recovers it as the chosen space of eigenfunctions is modified.
CHAPTER 2: How to Determine \( u_{NM} \) by extending the boundary data, \( g_N \)

For the problem on the unit disk, let \( \Omega \) be the unit disk, and let us assume a continuous current on the boundary. Assume the true expansion of the conductivity is:

\[
\sigma_{ij}(r, \theta) = \sum_{j=0}^{J} \sum_{l=0}^{L} (d_{ij} r^i \cos(j \theta) + e_{ij} r^i \sin(j \theta)).
\]

A Galerkin discretization employs a finite dimensional space to express \( u_{NM} \) as:

\[
u_{NM}(r, \theta) = \sum_{k=1}^{N} \sum_{l=1}^{M} (a_{kl} r^l \cos(k \theta) + b_{kl} r^l \sin(k \theta)) \in V_{NM} \subset H^1(\Omega).
\]

There are \( 2(N-1)M \) basis functions involved in this expansion of \( u_{NM} \) (see \( \phi_{kl} \) and \( \psi_{kl} \) below). The lower limits for the frequencies and the powers were restricted to 1 in order to ensure that \( V_{NM} \subset H^1(\Omega) \), and to work with discrete harmonic functions with mean zero. This will make the stiffness matrix for \( u_{NM} \) injective.

Let:

\[
\begin{align*}
\phi_{kl} &= r^l \cos(k \theta) \\
\psi_{kl} &= r^l \sin(k \theta),
\end{align*}
\]

then we have:

\[
\begin{align*}
\nabla \phi_{kl} &= l l^{l+1} \left( \cos \left( (k - \bar{k}) \theta \right) + \cos \left( (k + \bar{k}) \theta \right) \right) \\
&+ k \bar{k} r^{l+2} \left( \cos \left( (k - \bar{k}) \theta \right) - \cos \left( (k + \bar{k}) \theta \right) \right) \\
\nabla \psi_{kl} &= \frac{1}{2} l l^{l+1} \left( \sin \left( (k + \bar{k}) \theta \right) - \sin \left( (k - \bar{k}) \theta \right) \right) \\
&- \frac{1}{2} k \bar{k} r^{l+2} \left( \sin \left( (k - \bar{k}) \theta \right) + \sin \left( (k + \bar{k}) \theta \right) \right) \\
\end{align*}
\]

Using trigonometric formulas, we obtain the following expressions for gradient dot products:

\[
\begin{align*}
\nabla \phi_{kl} \cdot \nabla \phi_{kl} &= \frac{1}{2} l l^{l+1} \left( \cos \left( (k - \bar{k}) \theta \right) + \cos \left( (k + \bar{k}) \theta \right) \right) \\
&+ \frac{1}{2} k \bar{k} r^{l+2} \left( \cos \left( (k - \bar{k}) \theta \right) - \cos \left( (k + \bar{k}) \theta \right) \right) \\
\nabla \phi_{kl} \cdot \nabla \psi_{kl} &= \frac{1}{2} l l^{l+1} \left( \sin \left( (k + \bar{k}) \theta \right) - \sin \left( (k - \bar{k}) \theta \right) \right) \\
&- \frac{1}{2} k \bar{k} r^{l+2} \left( \sin \left( (k - \bar{k}) \theta \right) + \sin \left( (k + \bar{k}) \theta \right) \right) \\
\nabla \psi_{kl} \cdot \nabla \psi_{kl} &= \frac{1}{2} l l^{l+1} \left( \cos \left( (k - \bar{k}) \theta \right) + \cos \left( (k + \bar{k}) \theta \right) \right) \\
&+ \frac{1}{2} k \bar{k} r^{l+2} \left( \cos \left( (k - \bar{k}) \theta \right) - \cos \left( (k + \bar{k}) \theta \right) \right)
\end{align*}
\]

We leave out displaying the expression for \( \nabla \psi_{kl} \cdot \nabla \phi_{kl} \) since it is taken care of by \( \nabla \phi_{kl} \cdot \nabla \psi_{kl} \) through symmetry. Here, the trigonometric identities employed in the above gradient dot products were:

\[
\begin{align*}
\cos(k \theta) \cos(\bar{k} \theta) &= \frac{1}{2} \left( \cos \left( (k + \bar{k}) \theta \right) + \cos \left( (k - \bar{k}) \theta \right) \right) \\
\cos(k \theta) \sin(\bar{k} \theta) &= \frac{1}{2} \left( \sin \left( (k + \bar{k}) \theta \right) - \sin \left( (k - \bar{k}) \theta \right) \right) \\
\sin(k \theta) \sin(\bar{k} \theta) &= \frac{1}{2} \left( \cos \left( (k - \bar{k}) \theta \right) - \cos \left( (k + \bar{k}) \theta \right) \right) \\
\end{align*}
\]
For each of the dot products we get the following,
\[
\begin{align*}
\nabla \phi_{kl} \cdot \nabla \phi_{kl} &= (lr^{l-1} \cos(k\theta) \mathbf{u}_r - kr^{l-1} \sin(k\theta) \mathbf{u}_\theta) \cdot (lr^{l-1} \cos(k\theta) \mathbf{u}_r - kr^{l-1} \sin(k\theta) \mathbf{u}_\theta) \\
\nabla \phi_{kl} \cdot \nabla \psi_{kl} &= (lr^{l-1} \sin(k\theta) \mathbf{u}_r + kr^{l-1} \cos(k\theta) \mathbf{u}_\theta) \cdot (lr^{l-1} \cos(k\theta) \mathbf{u}_r - kr^{l-1} \sin(k\theta) \mathbf{u}_\theta) \\
\nabla \psi_{kl} \cdot \nabla \psi_{kl} &= (lr^{l-1} \sin(k\theta) \mathbf{u}_r + kr^{l-1} \cos(k\theta) \mathbf{u}_\theta) \cdot (lr^{l-1} \sin(k\theta) \mathbf{u}_r + kr^{l-1} \cos(k\theta) \mathbf{u}_\theta)
\end{align*}
\]
\[\mathbf{u}_r \text{ and } \mathbf{u}_\theta \text{ are components of a unit normal vector on the boundary.}\]

Let:
\[
\begin{align*}
\nabla \phi_{kl} \cdot \nabla \phi_{kl} & \text{ corresponds to } A_{kl}^{k, l} \\
\nabla \phi_{kl} \cdot \nabla \psi_{kl} & \text{ corresponds to } B_{kl}^{k, l} \\
\nabla \psi_{kl} \cdot \nabla \psi_{kl} & \text{ corresponds to } C_{kl}^{k, l}
\end{align*}
\]
In order to construct the Neumann matrix associated with \( \int_{\Omega} \sigma_{ij} \nabla u_{NM} \cdot \nabla v_{NM} \, dx \), note that one can see that:
\[
\begin{align*}
\nabla u_{NM} \cdot \nabla v_{NM} &= \left( \sum_{k=1}^{N} \sum_{l=1}^{M} (a_{kl} \nabla \phi_{kl} + b_{kl} \nabla \psi_{kl}) \right) \cdot \left( \sum_{k=1}^{N} \sum_{l=1}^{M} (a_{kl} \nabla \phi_{kl} + b_{kl} \nabla \psi_{kl}) \right) \\
&= \sum_{k=1}^{N} \sum_{l=1}^{M} \sum_{k'=1}^{N} \sum_{l'=1}^{M} (a_{kl} a_{k'l'} \nabla \phi_{kl} \cdot \nabla \phi_{k'l'} + b_{kl} b_{k'l'} \nabla \psi_{kl} \cdot \nabla \psi_{k'l'}) \\
&= \sum_{k=1}^{N} \sum_{l=1}^{M} \left( a_{kl} a_{k'l'} \nabla \phi_{kl} \cdot \nabla \phi_{k'l'} + b_{kl} b_{k'l'} \nabla \psi_{kl} \cdot \nabla \psi_{k'l'} \right)
\end{align*}
\]
as the sum of dot products of gradients of basis functions \( \left[ \begin{array}{c} \phi_{kl} \\ \psi_{kl} \end{array} \right]_{k=1, l=1}^{k=N, l=M} \) for \( u_{NM} \) and \( \left[ \begin{array}{c} \phi_{kl} \\ \psi_{kl} \end{array} \right]_{k=1, l=1}^{k=N, l=M} \) for \( v_{NM} \) respectively. To represent each combination in the dot products define the following integrals:

\[
\begin{align*}
A_{kl}^{(c), ji} &= \int_{0}^{1} \int_{0}^{2\pi} r^i \cos(j\theta) \left[ \frac{1}{2} l \tilde{r}^{l+1} + \frac{1}{2} k \tilde{r}^{l+1-2} \left( \cos((k - \tilde{k})\theta) + \cos((k + \tilde{k})\theta) \right) \right] d\theta dr \\
B_{kl}^{(c), ji} &= \int_{0}^{1} \int_{0}^{2\pi} r^i \sin(j\theta) \left[ \frac{1}{2} l \tilde{r}^{l+1} + \frac{1}{2} k \tilde{r}^{l+1-2} \left( \cos((k - \tilde{k})\theta) + \cos((k + \tilde{k})\theta) \right) \right] d\theta dr \\
C_{kl}^{(c), ji} &= \int_{0}^{1} \int_{0}^{2\pi} r^i \cos(j\theta) \left[ \frac{1}{2} l \tilde{r}^{l+1} + \frac{1}{2} k \tilde{r}^{l+1-2} \left( \sin((k + \tilde{k})\theta) - \sin((k - \tilde{k})\theta) \right) \right] d\theta dr \\
&\quad \quad + \int_{0}^{1} \int_{0}^{2\pi} r^i \sin(j\theta) \left[ \frac{1}{2} l \tilde{r}^{l+1} + \frac{1}{2} k \tilde{r}^{l+1-2} \left( \sin((k + \tilde{k})\theta) + \sin((k - \tilde{k})\theta) \right) \right] d\theta dr
\end{align*}
\]
\[
\begin{align*}
\mathbf{B}^{(s)}_{k,l,kl} &= \int_0^{2\pi} \int_0^1 r^i \sin(j\theta) \left[ \frac{1}{2} \ln r^{l+l-2} \left( \sin((k + \bar{k})\theta) - \sin((k - \bar{k})\theta) \right) \right] d\theta dr \\
\mathbf{C}^{(c)}_{k,l,kl} &= \int_0^{2\pi} \int_0^1 r^i \cos(j\theta) \left[ \frac{1}{2} \ln r^{l+l-2} \left( \cos((k - \bar{k})\theta) - \cos((k + \bar{k})\theta) \right) \right] d\theta dr \\
\mathbf{C}^{(s)}_{k,l,kl} &= \int_0^{2\pi} \int_0^1 r^i \sin(j\theta) \left[ \frac{1}{2} \ln r^{l+l-2} \left( \cos((k - \bar{k})\theta) - \cos((k + \bar{k})\theta) \right) \right] d\theta dr
\end{align*}
\]

These equations are further simplified below by applying the orthogonality of sinusoids at different frequencies.

\[
\begin{align*}
\mathbf{A}^{(c)}_{k,l,kl} &= \left( \frac{\pi}{i + l + \bar{l}} \right) \left[ \frac{1}{2} \ln r^{l+l-2} \left( \delta_{i,|k-\bar{k}|} + \delta_{i,(k+\bar{k})} \right) \right] \\
\mathbf{A}^{(s)}_{k,l,kl} &= 0 \\
\mathbf{B}^{(c)}_{k,l,kl} &= 0 \\
\mathbf{B}^{(s)}_{k,l,kl} &= \left( \frac{\pi}{i + l + \bar{l}} \right) \left[ \frac{1}{2} \ln r^{l+l-2} \left( \delta_{i,(k+\bar{k})} - \text{sign}(k - \bar{k}) \delta_{i,|k-\bar{k}|} \right) \right] \\
\mathbf{C}^{(c)}_{k,l,kl} &= \left( \frac{\pi}{i + l + \bar{l}} \right) \left[ \frac{1}{2} \ln r^{l+l-2} \left( \delta_{i,|k-\bar{k}|} - \delta_{i,(k+\bar{k})} \right) \right] \\
\mathbf{C}^{(s)}_{k,l,kl} &= 0
\end{align*}
\]

The superscripts (c) and (s) denote integrals with cosine and sine respectively. The Dirac deltas are used to reflect the orthogonality of the sinusoids at different frequencies. Also note the use of the absolute value to indicate that the integration results for the cosine terms do not depend on the sign of the difference between \(k\) and \(\bar{k}\). The even property of the cosine function makes this so.

Now we sum out the \(\sigma\) indices, \(i\) and \(j\).
\[
\begin{align*}
A_{k,l,kl} &= \sum_{j=0}^{J} \sum_{i=0}^{I} (d_{ji} A_{k,l,kl}^{(c),ji}) + \sum_{j=0}^{J} \sum_{i=0}^{I} (e_{ji} A_{k,l,kl}^{(s),ji}) \\
B_{k,l,kl} &= \sum_{j=0}^{J} \sum_{i=0}^{I} (d_{ji} B_{k,l,kl}^{(c),ji}) + \sum_{j=0}^{J} \sum_{i=0}^{I} (e_{ji} B_{k,l,kl}^{(s),ji}) \\
C_{k,l,kl} &= \sum_{j=0}^{J} \sum_{i=0}^{I} (d_{ji} C_{k,l,kl}^{(c),ji}) + \sum_{j=0}^{J} \sum_{i=0}^{I} (e_{ji} C_{k,l,kl}^{(s),ji})
\end{align*}
\]

There possible schemes by which the matrices can be ordered during coding. We fix the \(l\) and \(\bar{l}\) indices at a go while varying the \(k\) and \(\bar{k}\) indices. This creates the sub-matrices in the \(k\) and \(\bar{k}\) indices. The following annotated matrix provides an illustration.

\[
\begin{pmatrix}
\begin{array}{cccc}
\bar{k} = 1, k = 1 & \cdots & \bar{k} = 1, k = N \\
\vdots & \ddots & \vdots \\
\bar{k} = N, k = 1 & \cdots & \bar{k} = N, k = N \\
\bar{k} = 1, k = 1 & \cdots & \bar{k} = 1, k = N \\
\vdots & \ddots & \vdots \\
\bar{k} = N, k = 1 & \cdots & \bar{k} = N, k = N \\
\end{array}
\end{pmatrix}
\]

We are now ready to construct the stiffness matrix associated with

\[
\int_{\Omega} \sigma_{NM} \nabla u_{NM} \cdot \nabla v_{NM} \, dx
\]

as:

\[
K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix},
\]

Where the entries and the unknowns are ordered using the scheme described above.

From now on, we will use the same letters to represent the functions and their lists of expansion coefficients. Therefore, the integral is: \(U^T K U\), where

\[
\left( [a_{kl}]_{k=1,l=1}^{k=N,l=M} , [b_{kl}]_{k=1,l=1}^{k=N,l=M} \right) \]

are the coefficient entries of \(U\).

Next, we need to incorporate the Dirichlet boundary condition. Imposing the condition that \(u_{NM}(1, \theta) = g_{NM}(\theta)\), we use the fact that \(g_{NM}(\theta)\) is independent of \(r\).

\[
\begin{align*}
u_{NM}(r, \theta) &= \sum_{k=1}^{N} \sum_{l=1}^{M} (a_{kl} r^l \cos(k \theta) + b_{kl} r^l \sin(k \theta)) \\
u_{NM}(1, \theta) &= \sum_{k=1}^{N} \sum_{l=1}^{M} (a_{kl} \cos(k \theta) + b_{kl} \sin(k \theta)) \quad \Rightarrow \quad g_k^c = \sum_{l=1}^{M} (a_{kl}) \\
g_k^s = \sum_{l=1}^{M} (b_{kl}) \\
g_{NM}(\theta) &= \sum_{k=1}^{N} (g_k^c \cos(k \theta) + g_k^s \sin(k \theta))
\end{align*}
\]
These give us extra equations to use to augment the stiffness matrix, $K$. To do this, notice that the $k$ index is fixed while the $l$ index is summed out in the expressions for $g_k^c$ and $g_k^s$.

To do this summation in matrix form, we create a new matrix, $W^T$, as:

$$W^T = \begin{bmatrix}
I_{N \times N} & I_{N \times N} & \ldots & I_{N \times N} \\
I_{N \times N} & I_{N \times N} & \ldots & I_{N \times N} \\
0 & 0 & \ldots & 0 \\
\text{Cosine part} & \text{Sine part}
\end{bmatrix}
$$

where $I_{N \times N}$ is the $N \times N$ identity matrix and 0 represents an $N \times N$ zero matrix.

In addition, we reconsider the equations,

$$0 = -\int_{\Omega} \nabla \cdot (\sigma_{ij} \nabla u_{NM}) v_{NM} \, dx \quad \text{and} \quad \int_{\Omega} \nabla \cdot (\sigma_{ij} \nabla u_{NM}) v_{NM} \, dx = \int_{\partial \Omega} (\sigma_{ij} \nabla u_{NM}) v_{NM} \, d\mathbf{S},$$

with $\int_{\Omega} \nabla \cdot (\sigma_{ij} \nabla u_{NM}) \, dx = V^T K U$, $\int_{\partial \Omega} (\sigma_{ij} \nabla u_{NM}) v_{NM} \, d\mathbf{S} = V^T \Xi$, and $V = U$.

The arrangement of entries in $U$ follows the arrangement chosen for the rows of $K$ in that $l$ is fixed while $k$ is varied to match the rows of $K$. Here, we are using a minimization method with Lagrange multipliers, $\Xi$.

After augmenting $K$, we can create the following system:

$$\begin{bmatrix}
K & W \\
W^T & 0
\end{bmatrix}
\begin{bmatrix}
U \\
\Xi
\end{bmatrix} =
\begin{bmatrix}
0 \\
G
\end{bmatrix}.
$$

The first part, $K U + W \Xi = 0$, accounts for the equation, $\int_{\Omega} \nabla \cdot (\sigma_{ij} \nabla u_{NM}) v_{NM} \, dx = 0$ in $\Omega$. The second part, $W^T U = G$, incorporates the boundary conditions. $G$ in $\begin{bmatrix}
0 \\
G
\end{bmatrix}$ is the diagonal matrix, $\text{diag}\left(\begin{bmatrix} g_k^c \\ g_k^s \end{bmatrix}_{k=1}^{k=N} \right)$. (The $\text{diag}(\ )$ command in MATLAB inserts entries of a vector into the diagonal of a matrix. It is used here to represent that process). The size of $W$ is $2N \times 2NM$, and the size of is $G$ is $2N \times 2N$. Note that for a harmonic solution, $U$ in $\begin{bmatrix}
U \\
\Xi
\end{bmatrix}$ is also a matrix containing two diagonal matrices. The entries are made up of

$$\begin{bmatrix}
\text{diag}(\begin{bmatrix} a_{kl} \end{bmatrix}_{k=1,l=1}^{k=N,l=M}) & 0 \\
0 & \text{diag}(\begin{bmatrix} b_{kl} \end{bmatrix}_{k=1,l=1}^{k=N,l=M})
\end{bmatrix}.
$$

When $u_{NM}$ is not harmonic, $U$ becomes a combination of harmonic and non-harmonic parts and the coefficient matrix will no longer be diagonal. It will contain off-diagonal entries.
CHAPTER 3: How to Determine $S_{2N \times 2N}$:

The entries of the matrix, $S_{2N \times 2N}$, are obtained from the entries of $U^T K U$. $K$ contains the value for the integral of each eigenfunction involved in creating $S$. The subscript, $2N \times 2N$, refers to the size of the resulting matrix. Let us get a heuristic for computing this form of $S_{2N \times 2N}$ after computing $U$. We recall the system derived in the later part of chapter 2 and use it to re-express $S_{2N \times 2N}$:

\[
\begin{align*}
\{ KU + WZ & = 0 \\
W^T U + 0 & = G 
\end{align*}
\]

\[
\Rightarrow \begin{cases} 
KU = -WZ \text{ and } U = -K^{-1}WZ \\
-W^T K^{-1}WZ = G \text{ and } Z = -(W^T K^{-1}W)^{-1}G \text{ and } U = K^{-1}W(W^T K^{-1}W)^{-1}G 
\end{cases}
\]

$S_{2N \times 2N} = (U^T KU) = (K^{-1}W(W^T K^{-1}W)^{-1}G)^T K(K^{-1}W(W^T K^{-1}W)^{-1}G)$

Making $G = I_{2N \times 2N}$ causes cancellations to occur due to the fact that $K$ being symmetric implies that $K^{-1}$ and $Z$ will also be symmetric. We have:

\[
(S_{2N \times 2N}) = (K^{-1}W(W^T K^{-1}W)^{-1}I_{N \times N})^T K(K^{-1}W(W^T K^{-1}W)^{-1}I_{N \times N})
\]

\[
= (K^{-1}W(W^T K^{-1}W)^{-1})^T (W(W^T K^{-1}W)^{-1}) \quad \text{cancelled } K K^{-1}
\]

\[
= ((W^T K^{-1}W)^{-1}) W^T K^{-1} (W(W^T K^{-1}W)^{-1}) \quad \text{applied the transpose and used the symmetry of } K^{-1}
\]

\[
= ((W^T K^{-1}W)^{-1}) (W^T K^{-1}W) (W^T K^{-1}W)^{-1} \quad \text{rearranged}
\]

\[
= (W^T K^{-1}W)^{-1} \quad \text{by simplification}
\]

If $g$ is not the identity matrix, it will appear in the formula. To modify the formula, note that:

\[
S_{2N \times 2N} = G((W^T K^{-1}W)^{-1}) (W^T K^{-1}W) (W^T K^{-1}W)^{-1}G.
\]

Since $G$ is diagonal, applying commutation of $g$ with $2N \times 2N$ matrices after repeating the above simplifications, we get:

\[
S_{2N \times 2N} = (W^T K^{-1}W)^{-1} G^2
\]
CHAPTER 4: How to get $\sigma_R(r, \theta)$

We have two ways to determine $\sigma_R$. Let $\sigma_{jl}(r, \theta)$ have the form,

$$\sigma_{jl}(r, \theta) = \sum_{j=0}^{J} \sum_{i=0}^{I} (d_{ji} r^i \cos(j \theta) + e_{ji} r^i \sin(j \theta)),$$

where $\sigma_{jl}$ is the truncated expansion of $\sigma$.

To obtain $\sigma_{jl}$, $\sigma$ is expanded using orthogonal polynomials in the radial direction and sinusoids in the angular direction. Then the coefficients from each polynomial are rearranged to represent the expansion in the non-orthogonal basis used in this report’s discussion.

If you linearize for $\sigma_{jl} \approx 1 + \tilde{\sigma}_{jl} = \sigma_R$, there is a relationship between the entries in the coefficient matrix, $K$, and the vector of $\sigma_R$’s coefficients: $K\Sigma = S_{2N \times 2N}$. $S_{2N \times 2N}$ is unfolded into a column vector array. The first method we can use is to make sure that we have the same number of unknowns for $\Sigma$ as the number of equations for $S_{2N \times 2N}$.

First, note that when $\sigma = 1$, our PDE problem becomes the Laplace’s equation:

$$\begin{align*}
-\Delta u &= 0 \text{ in } \Omega \\
u &= g \text{ in } \partial\Omega
\end{align*}$$

A linear combination of harmonic functions solves this equation. Using polar coordinates on the unit disk, we express the solution as:

$$\begin{align*}
u &= \sum_{k=0}^{\infty} r^k (a_k \cos(k \theta) + b_k \sin(k \theta)) \\
g &= \sum_{k=0}^{\infty} (a_k \cos(k \theta) + b_k \sin(k \theta))
\end{align*}$$

Here, $u$ is the harmonic extension of $g$. Notice that the eigenfunctions involved are: $\{1, \{\cos(k \theta), \sin(k \theta)\}\}_{1}^{\infty}$.

In linearizing the discrete Dirichlet to Neumann map, using $\sigma_{jl} \approx 1 + \tilde{\sigma}_{jl}$, we get:

$$\begin{align*}
\left( S_{2N \times 2N}^{(1)} u_{NM}, v_{NM} \right) &= \int_{\Omega} \sigma_{jl}^{(1)} \nabla u_{NM} \cdot \nabla v_{NM} \, dx, & \sigma_{jl}^{(1)} = 1 \text{ on } \partial\Omega \\
\left( S_{2N \times 2N}^{(2)} u_{NM}, v_{NM} \right) &= \left( S_{2N \times 2N}^{(1)} u_{NM}, v_{NM} \right) + \int_{\Omega} \tilde{\sigma}_{jl} \nabla u_{NM} \cdot \nabla v_{NM} \, dx, & \tilde{\sigma}_{jl} = 0 \text{ on } \partial\Omega \\
\left( S_{2N \times 2N} u_{NM}, v_{NM} \right) &= \int_{\Omega} \sigma_{jl} \nabla u_{NM} \cdot \nabla v_{NM} \, dx \approx \int_{\Omega} \sigma_{R} \nabla u_{NM} \cdot \nabla v_{NM} \, dx = \left( S_{2N \times 2N}^{(2)} u_{NM}, v_{NM} \right)
\end{align*}$$

Let:
\[
\begin{align*}
\phi_{kl} &= \cos(k\theta) \Rightarrow E\phi_{kl} = r^l \cos(k\theta) \\
\psi_{kl} &= \sin(k\theta) \Rightarrow E\psi_{kl} = r^l \sin(k\theta),
\end{align*}
\]

where \( \phi_{kl} \) and \( \psi_{kl} \) are basis functions in the expansions for \( g \).

We use \( E\phi_{kl} \) and \( E\psi_{kl} \) to represent the extensions of \( \phi_{kl} \) and \( \psi_{kl} \). Recall that:

\[
\begin{align*}
\nabla &\cdot \nabla (1)\phi_{kl} = (l r^{l-1} \cos(k\theta) u_r - k r^{l-1} \sin(k\theta) u_0) \cdot (l r^{l-1} \cos(k\theta) u_r - k r^{l-1} \sin(k\theta) u_0) \\
\nabla &\cdot \nabla (1)\psi_{kl} = (l r^{l-1} \cos(k\theta) u_r - k r^{l-1} \sin(k\theta) u_0) \cdot (l r^{l-1} \sin(k\theta) u_r + k r^{l-1} \cos(k\theta) u_0) \\
\nabla &\cdot \nabla (1)\psi_{kl} = (l r^{l-1} \sin(k\theta) u_r + k r^{l-1} \cos(k\theta) u_0) \cdot (l r^{l-1} \sin(k\theta) u_r + k r^{l-1} \cos(k\theta) u_0)
\end{align*}
\]

\( u_r \) and \( u_0 \) are components of a unit normal vector on the boundary.

After setting \( k = l \) and \( \bar{k} = \bar{l} \), for the harmonic case, we use \( E(1)\phi_{kl} \) and \( E(1)\psi_{kl} \) to represent the harmonic extensions of \( \phi_{kl} \) and \( \psi_{kl} \). Replace \( l \) with \( k \) and \( \bar{l} \) with \( \bar{k} \) in the above inner products. Then factor them to get:

\[
\begin{align*}
\nabla &\cdot \nabla (1)\phi_{k} = k \bar{k} r^{k+k-1} (\cos(k\theta) u_r - \sin(k\theta) u_0) \cdot (\cos(k\theta) u_r - \sin(k\theta) u_0) \\
\nabla &\cdot \nabla (1)\psi_{k} = k \bar{k} r^{k+k-1} (\cos(k\theta) u_r - \sin(k\theta) u_0) \cdot (\sin(k\theta) u_r + \cos(k\theta) u_0) \\
\nabla &\cdot \nabla (1)\psi_{k} = k \bar{k} r^{k+k-1} (\sin(k\theta) u_r + \cos(k\theta) u_0) \cdot (\sin(k\theta) u_r + \cos(k\theta) u_0)
\end{align*}
\]

\[
\begin{align*}
\nabla &\cdot \nabla (1)\phi_{k} = k \bar{k} r^{k+k-2} \left( \cos(k\theta) \cos(k\bar{k}) + \sin(k\theta) \sin(k\bar{k}) \right) \\
\nabla &\cdot \nabla (1)\psi_{k} = k \bar{k} r^{k+k-2} \left( \cos(k\theta) \sin(k\bar{k}) - \sin(k\theta) \cos(k\bar{k}) \right) \\
\nabla &\cdot \nabla (1)\psi_{k} = k \bar{k} r^{k+k-2} \left( \sin(k\theta) \sin(k\bar{k}) + \cos(k\theta) \cos(k\bar{k}) \right)
\end{align*}
\]

After applying the sum and difference trigonometric identities, we obtain:

\[
\begin{align*}
\nabla &\cdot \nabla (1)\phi_{k} = k \bar{k} r^{k+k-2} \left( \cos((k - \bar{k})\theta) \right) \\
\nabla &\cdot \nabla (1)\psi_{k} = k \bar{k} r^{k+k-2} \left( \sin((k - \bar{k})\theta) \right) \\
\nabla &\cdot \nabla (1)\psi_{k} = k \bar{k} r^{k+k-2} \left( \cos((k - \bar{k})\theta) \right)
\end{align*}
\]

As mentioned earlier, first method for finding \( \sigma_R \) is to match the number of unknowns to the number of equations that involve eigenfunctions used to create \( u_{NM} \). To accomplish this, we, first, apply the symmetry obtained from harmonic solutions to remove all linear dependence among the eigenfunctions. Then we force \( \sigma_R \) to satisfy the unit boundary condition.

Recall that \( g_N \) is the boundary data. This means that its basis should coincide with the basis used to describe the voltage distribution on the boundary electrode(s). Since \( u_{NM} \) is the harmonic extension of \( g_N \), the number of eigenfunctions for \( u_{NM} \) is controlled...
by the number for \( g_N \). So we can deduce the number of eigenfunctions for expanding \( u_{NM} \) by studying the size of the subspace that generates \( g_N \). This then determines the number of equations involved in creating the entries of the \( S_{2N \times 2N} \) matrix along with the number of basis functions for expanding \( \sigma_R \).

Let \( j = (k - \bar{k}) \) with \( k\bar{k}r^{k+\bar{k}-2} = \bar{k}(\bar{k} + j)r^{2\bar{k}+j-2} \). Let \( V_{2q} \) be the space from which the boundary electrode voltage distribution is generated. Given \( V_{2q} = \text{span}\{\cos(k\theta), \sin(k\theta)\}_{k=1}^{q} \), we define the space for creating \( S_{2N \times 2N} \) as:

\[
\Pi_q = \text{span}\{\nabla E(1)\phi_k \cdot \nabla E(1)\psi_k | \phi_k, \psi_k \in V_{2q}\}
\]

\[
= \text{span} \left\{ (r^{2(k-1)+j} \cos(j\theta), r^{2(\bar{k}-1)+j} \sin(j\theta))_{\bar{k}=1}^{q-j} \right\}_{j=0}^{q-1}
\]

We obtain the above limits by observing that \( 1 \leq k \leq q \) and \( 1 \leq \bar{k} \leq q \) imply that the maximum value for \( j \) is \( q - 1 \). (Note that we excluded \( k, \bar{k} = 0 \) because we are neither including the zero function nor functions with unbounded gradients on the unit disk when we select \( \phi_k \) and \( \psi_{\bar{k}} \).) One might guess that the minimum should be \( 1 - q \). However, sinusoids with the same absolute frequency value are linearly dependent. So, we restrict \( j \) to be positive: \( q - 1 \geq j \geq 0 \). Combining this restriction with the definition that \( j = (k - \bar{k}) \) then results in the above limits in the definition of \( \Pi_q \).

By construction, \( \text{dim}(V_{2q}) = 2q \) and we may heuristically expect

\[
\text{dim}(\Pi_q) = q(2q + 1)
\]

That is, we might think there are \( q \) powers and \( 2q + 1 \) sinusoids combined to span \( \Pi_q \). Again, this is not correct because of the terms from \( \nabla E(1)\phi_i \cdot \nabla E(1)\psi_j \) that are linearly dependent. Studying the enumeration of the basis for \( \Pi_q \) more closely one should realize that there are \( \frac{q^2}{2} + \frac{q}{2} \) cases containing cosine, and \( \frac{q^2}{2} + \frac{q}{2} \) cases containing sine. We then remove the \( q \) cases containing \( \sin(0) \) from the sum of all the number of cases to get a total of \( q^2 \) basis functions. Pictorially, what happened in the count was:
Because we fixed the value of $\sigma_R$ on the boundary, we cannot assign $q^2$ to the dimension of the space used to create $\sigma_R$. We need to refine it further by enforcing the boundary condition on $\sigma_R$. Look again at the formula for $\sigma_R$ and notice that:

$$\sigma_R(r, \theta) = \sum_{j=0}^{q-1} \sum_{i=1}^{q-j} (d_{ji}r^{2(i-1)+j} \cos(j\theta)) + \sum_{j=1}^{q} \sum_{i=1}^{q-j} (e_{ji}r^{2(i-1)+j} \sin(j\theta))$$

This implies that we get the following constraints:

$$0 = \sum_{i=1}^{q-j} (d_{ji}) \quad \text{for} \quad 1 \leq j \leq q - j$$

$$1 = \sum_{i=1}^{q-j} (d_{ji}) \quad \text{for} \quad j = 0$$

$$0 = \sum_{i=1}^{q-j} (e_{ji}) \quad \text{for} \quad 1 \leq j \leq q - j$$

The constraints provide $(2q - 1)$ equations. We subtract this number from $q^2$ to give the dimension, $\text{dim}(\Pi_{q,0}) = (q - 1)^2$, for the space used to expand $\sigma_R$. To form $\Pi_{q,0}$, we
select \((q - 1)^2\) eigenfunctions from \(\Pi_q\). Using \(N\) as the number of basis functions for \(g_N\), we set \(q = N = N_\sigma + 1\). This means that we will have \(\sigma\) expanded as:

\[
\sigma_R (r, \theta) = \sum_{j=0}^{N-1} \sum_{i=1}^{N-j} (d_{ij} r^{2(i-1)+j} \cos(j \theta) + \sum_{j=1}^{N-1} \sum_{i=1}^{N-j} (e_{ij} r^{2(i-1)+j} \sin(j \theta)).
\]

\((N - 1)^2\) is the number of basis functions required to make the map from \(\sigma_R\) to \(S_{2N \times 2N}\) injective.

Now we can start constructing the stiffness matrix for determining \(\sigma_R\). As usual, we proceed from the definition of \(S\) with \(k = l, \bar{k} = \bar{l},\) and \(j = (k - \bar{k})\). (Note that \(j\) is not the same index as \(j\) used in expanding \(\sigma_R\).) In order to construct the matrix associated, we follow the method outlined in chapter 2. Note that one can see that:

\[
\begin{align*}
\nabla u \cdot \nabla v &= (\sum_{k=1}^{N} (a_{kk} \nabla \phi_{kk} + b_{kk} \nabla \psi_{kk})) \cdot (\sum_{k=1}^{N} (a_{kk} \nabla \phi_{kk} + b_{kk} \nabla \psi_{kk})) \\
&= \sum_{k=1}^{N} \sum_{k=1}^{N} (a_{kk} \nabla \phi_{kk} + b_{kk} \nabla \psi_{kk}) \cdot (a_{kk} \nabla \phi_{kk} + b_{kk} \nabla \psi_{kk}) \\
&= \sum_{k=1}^{N} \sum_{k=1}^{N} \left( a_{kk} a_{kk} \nabla \phi_{kk} \cdot \nabla \phi_{kk} + b_{kk} b_{kk} \nabla \psi_{kk} \cdot \nabla \psi_{kk} + a_{kk} b_{kk} \nabla \phi_{kk} \cdot \nabla \psi_{kk} \right)
\end{align*}
\]

as the sum of dot products of gradients of basis functions \(\phi_{kk}\) for \(u_{NM}\) and \(\psi_{kk}\) for \(v_{NM}\). Let \(j \leq \bar{i} = 2(i - 1) + j \leq 2(N - j - 1) + j = 2N - j - 2\). To represent the sub-matrices obtained from each combination in the dot products define the following integrals:

\[
\begin{align*}
A^{(c),ij}_{\bar{k},k} &= \int_{0}^{2\pi} \int_{0}^{r} r^i \cos(j \theta) \, d\theta dr \\
&= \left[ \frac{1}{2} k \bar{k} \bar{r}^{k+\bar{k}-2} \left( \cos((k - \bar{k})\theta) + \cos((k + \bar{k})\theta) \right) + \frac{1}{2} \bar{k} \bar{k} \bar{r}^{k+\bar{k}-2} \left( \cos((k - \bar{k})\theta) - \cos((k + \bar{k})\theta) \right) \right] d\theta dr \\
A^{(s),ij}_{\bar{k},k} &= \int_{0}^{2\pi} \int_{0}^{r} r^i \sin(j \theta) \, d\theta dr \\
&= \left[ \frac{1}{2} k \bar{k} \bar{r}^{k+\bar{k}-2} \left( \cos((k - \bar{k})\theta) + \cos((k + \bar{k})\theta) \right) + \frac{1}{2} \bar{k} \bar{k} \bar{r}^{k+\bar{k}-2} \left( \cos((k - \bar{k})\theta) - \cos((k + \bar{k})\theta) \right) \right] d\theta dr \\
B^{(c),ij}_{\bar{k},k} &= \int_{0}^{2\pi} \int_{0}^{r} r^i \cos(j \theta) \, d\theta dr \\
&= \left[ \frac{1}{2} k \bar{k} \bar{r}^{k+\bar{k}-2} \left( \sin((k + \bar{k})\theta) - \sin((k - \bar{k})\theta) \right) + \frac{1}{2} \bar{k} \bar{k} \bar{r}^{k+\bar{k}-2} \left( \sin((k + \bar{k})\theta) + \sin((k - \bar{k})\theta) \right) \right] d\theta dr \\
B^{(s),ij}_{\bar{k},k} &= \int_{0}^{2\pi} \int_{0}^{r} r^i \sin(j \theta) \, d\theta dr
\end{align*}
\]
\[
B_{k,k}^{(s),i,j} = \int_0^{2\pi} \int_0^r r^i \sin(j\theta) \left[ \frac{1}{2} k \bar{k} r^{k+\bar{k}-2} \left( \sin \left( (k + \bar{k})\theta \right) - \sin \left( (k - \bar{k})\theta \right) \right) \right] d\theta dr
\]

\[
C_{k,k}^{(c),i,j} = \int_0^{2\pi} \int_0^r r^i \cos(j\theta) \left[ \frac{1}{2} k \bar{k} r^{k+\bar{k}-2} \left( \cos \left( (k + \bar{k})\theta \right) - \cos \left( (k - \bar{k})\theta \right) \right) \right] d\theta dr
\]

These equations are further simplified below by applying the orthogonality of sinusoids at different frequencies.

\[
A_{k,k}^{(c),i,j} = \left( \frac{\pi}{i + k + \bar{k}} \right) \left[ \frac{1}{2} k \bar{k} r^{k+\bar{k}-2} \left( \delta_{i,|k-\bar{k}|} + \delta_{i,(k+\bar{k})} \right) \right] + \left[ \frac{1}{2} k \bar{k} r^{k+\bar{k}-2} \left( \delta_{i,|k-\bar{k}|} - \delta_{i,(k+\bar{k})} \right) \right]
\]

\[
A_{k,k}^{(s),i,j} = 0
\]

\[
B_{k,k}^{(c),i,j} = \left( \frac{\pi}{i + k + \bar{k}} \right) \left[ \frac{1}{2} k \bar{k} r^{k+\bar{k}-2} \left( \delta_{i,(k+\bar{k})} - \text{sign}(k - \bar{k})\delta_{i,|k-\bar{k}|} \right) \right] - \left[ \frac{1}{2} k \bar{k} r^{k+\bar{k}-2} \left( \text{sign}(k - \bar{k})\delta_{i,|k-\bar{k}|} + \delta_{i,(k+\bar{k})} \right) \right]
\]

\[
C_{k,k}^{(s),i,j} = 0
\]

The superscripts (c) and (s) denote integrals with cosine and sine respectively. Now, return and replace \(i\) with \(2(i - 1) + j\) in the right-hand expressions. Then, replace \(i\) with \(i\) as an index in the left-hand side of each of the above equations.

Below is a scheme by which sub-matrices can be ordered during coding. While going along a row, we fix the \(j\) index at a go while varying the \(i\) index. While going along a column, we fix the \(\bar{k}\) index at a go while varying the \(k\) index. The following annotated matrix provides an illustration.
Unfold the arrays so that $i$ and $j$ vary along the rows, and have $k$ and $\tilde{k}$ vary down the columns. Also, let $A^c_{1}$ be the unfolded matrix when $\nabla E(1)\phi_k \cdot \nabla E(1)\phi_{\tilde{k}}$ combines with the cosine part of $\sigma_R$. Let $B^c_{1}$ be the unfolded matrix when $\nabla E(1)\phi_k \cdot \nabla E(1)\psi_{\tilde{k}}$ combines with the cosine part of $\sigma_R$. Let $B^c_{2}$ be the unfolded matrix when $\nabla E(1)\phi_k \cdot \nabla E(1)\psi_{\tilde{k}}$ combines with the cosine part of $\sigma_R$. Let $C^c_{1}$ be the unfolded matrix when $\nabla E(1)\psi_k \cdot \nabla E(1)\phi_{\tilde{k}}$ combines with the sine part of $\sigma_R$. Let $A^{s}_{1}$ be the unfolded matrix when $\nabla E(1)\phi_k \cdot \nabla E(1)\phi_{\tilde{k}}$ combines with the sine part of $\sigma_R$. Let $B^{s}_{1}$ be the unfolded matrix when $\nabla E(1)\psi_k \cdot \nabla E(1)\psi_{\tilde{k}}$ combines with the sine part of $\sigma_R$. Let $B^{s}_{2}$ be the unfolded matrix when $\nabla E(1)\phi_k \cdot \nabla E(1)\phi_{\tilde{k}}$ combines with the sine part of $\sigma_R$. Let $C^{s}_{1}$ be the unfolded matrix when $\nabla E(1)\psi_k \cdot \nabla E(1)\psi_{\tilde{k}}$ combines with the sine part of $\sigma_R$. We are now ready to construct the stiffness matrix associated with $\int_{\Omega} \sigma_{R} \nabla u_{NM} \cdot \nabla v_{NM} \, dx$ as:

$$
K = \begin{bmatrix}
A^c_{1} & A^{s}_{1} \\
B^c_{1} & B^{s}_{1} \\
B^c_{2} & B^{s}_{2} \\
C^c_{1} & C^{s}_{1}
\end{bmatrix}
$$

Where the entries and the unknowns are ordered using the scheme described above.

Next, we need to remove the constant term and incorporate the zero boundary condition.

Imposing the conditions that $\bar{\sigma}(1, \theta) = 0$ and $\sigma_R(1, \theta) = 1$, we use the fact that $\sigma_R(1, \theta)$ is independent of $r$. 
\[
\begin{aligned}
\sigma_R (r, \theta) &= \sum_{j=0}^{N-1} \sum_{i=1}^{N-j} \left( d_{ji} r^{2(j-1)+j} \cos(j \theta) \right) + \sum_{j=1}^{N-1} \sum_{i=1}^{N-j} \left( e_{ji} r^{2(j-1)+j} \sin(j \theta) \right) \\
\sigma_R (1, \theta) &= \sum_{j=0}^{N-1} \sum_{i=1}^{N-j} \left( d_{ji} \cos(j \theta) \right) + \sum_{j=1}^{N-1} \sum_{i=1}^{N-j} \left( e_{ji} \sin(j \theta) \right)
\end{aligned}
\]

These give us extra equations to use to augment the stiffness matrix, \( K \). To do this, notice that the \( j \) index is fixed while the \( i \) index is summed out in the last two expressions. We will do this summation in matrix format. Define the vector, \( \mathbf{1}_{1 \times (N-j)} = [1, \ldots, 1] \), of one’s repeated \( N - j \) times. Then create a new matrix, \( \mathbf{W}^T \), as:

\[
\mathbf{W}^T = \begin{bmatrix}
\mathbf{W}^{(c)^T} & \mathbf{0} \\
\mathbf{0} & \mathbf{W}^{(s)^T}
\end{bmatrix}
\]

\[
\mathbf{W}^{(c)^T} = \begin{bmatrix}
\begin{array}{cccc}
1 & 1 & \ldots & 1 \\
& & & \\
& & & \\
& & & \\
\end{array}
\end{bmatrix}_{1 \times (N-j)}
\]

\( Cosine \ part \)

\[
\mathbf{W}^{(s)^T} = \begin{bmatrix}
\begin{array}{cccc}
1 & 2 & \ldots & N-1 \\
& & & \\
& & & \\
& & & \\
\end{array}
\end{bmatrix}_{1 \times (N-j)}
\]

\( Sine \ part \)

Next, we augment \( K \) with \( \mathbf{W}^T \).
We obtain the following constrained least squares system:

\[
\begin{bmatrix}
K^T K & W \\
W^T & 0
\end{bmatrix}
\begin{bmatrix}
\Sigma \\
\Gamma
\end{bmatrix}
= \begin{bmatrix}
K^T S_{2N \times 2N} \\
0 \\
\end{bmatrix}.
\]

The first part, \( K \Sigma = S_{2N \times 2N} \), accounts for \( \int_{\Omega} \sigma_R \nabla u_{NM} \cdot \nabla v_{NM} \, dx \) in \( \Omega \). The second part, \( W^T \Sigma = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), incorporates the boundary conditions enforced upon \( \tilde{\sigma} \).

For the least squares method, replace the limits of \( i \) and \( j \) with arbitrary limits, \( J \) and \( I \). For example, let such that \( I = N \) and \( J = N - 1 \). Then, for those chosen limits, represent \( \sigma_R \) as:

\[
\sigma_R(r, \theta) = \sum_{i=1}^{N} \sum_{j=0}^{N-1} (\tilde{d}_{ji} r^i \cos(j \theta)) h(i + j - \tilde{N}(i)) + \sum_{i=1}^{N} \sum_{j=0}^{N-1} (\tilde{e}_{ji} r^i \sin(j \theta)) h(i + j - \tilde{N}(i))
\]

where,

\[
h(i + j - \tilde{N}(i)) = \begin{cases} 
0 & \text{for } j > \tilde{N}(i) - i \\
1 & \text{for } j \leq \tilde{N}(i) - i 
\end{cases}
\]

and

\[
\begin{cases} 
\tilde{d}_{ji} = \tilde{d}_{ji} h(i + j - \tilde{N}(i)) \\
\tilde{e}_{ji} = \tilde{e}_{ji} h(i + j - \tilde{N}(i))
\end{cases}
\]

Here, \( h(i + j - \tilde{N}(i)) \) is used to select the coefficient indices we want to use in expanding \( \tilde{\sigma} \). For the case when \( \tilde{N}(j) = N \), below is a scheme by which sub-matrices can be ordered during coding. While going along a row, we fix the \( i \) index at a go while varying the \( j \) index. While going along a column, we fix the \( \tilde{k} \) index at a go while varying the \( k \) index. If one does the specified integrals for \( K \) and replaces \( i \) with \( 2(i - 1) + j \) in the right-hand formulas, but re-labels the left-hand index as \( i \), the following annotated matrix provides an illustration for how to arrange entries.

\[
\begin{align*}
&k = 1, i = 1 \\
&\begin{pmatrix}
\tilde{k} = 1, j = 0 & \cdots & \tilde{k} = 1, j = N - 1 \\
\vdots & & \vdots \\
\tilde{k} = N, j = 0 & \cdots & \tilde{k} = N, j = N - 1
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
&k = 2 \\
&\begin{pmatrix}
\tilde{k} = 1, j = 0 & \cdots & \tilde{k} = 1, j = N - 1 \\
\vdots & & \vdots \\
\tilde{k} = N, j = 0 & \cdots & \tilde{k} = N, j = N - 1
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
&\ldots k = N \\
&\begin{pmatrix}
\tilde{k} = 1, j = 0 & \cdots & \tilde{k} = 1, j = N - 1 \\
\vdots & & \vdots \\
\tilde{k} = N, j = 0 & \cdots & \tilde{k} = N, j = N - 1
\end{pmatrix}
\end{align*}
\]

\text{Cosine part}
We are now ready to construct the stiffness matrix associated with $\int_\Omega \sigma_R \nabla u_{NM} \cdot \nabla v_{NM} \, dx$ as before, but the entries and the unknowns are ordered using the scheme described above. We also need to incorporate the zero boundary condition. Imposing the condition that $\sigma_1(1,\theta) = 0$, we use the fact that $\sigma_R(1,\theta)$ is independent of $r$.

\[
\sigma_R(r, \theta) = \sum_{j=0}^{N-1} \sum_{i=1}^{N} (d_j r^i \cos(j\theta)) + \sum_{j=1}^{N-1} \sum_{i=1}^{N} (e_j r^i \sin(j\theta))
\]

\[
\sigma_R(1, \theta) = \sum_{j=0}^{N-1} \sum_{i=1}^{N} (d_j \cos(j\theta)) + \sum_{j=1}^{N-1} \sum_{i=1}^{N} (e_j \sin(j\theta))
\]

\[
\begin{align*}
0 &= \sum_{i=1}^{N} (d_j) \text{ for } 1 \leq j \leq N - j \\
1 &= \sum_{i=1}^{N} (d_j) \text{ for } j = 0 \\
0 &= \sum_{i=1}^{N} (e_j) \text{ for } 1 \leq j \leq N - j
\end{align*}
\]

These give us extra equations to use to augment the stiffness matrix, $K$. To do this, notice that the $j$ index is fixed while the $i$ index is summed out in the last two expressions. We will do this summation in matrix format. However, we are going to fix $i$ first and vary $j$. Define the matrix, $I_{(N-1) \times (N-1)}$, as the identity matrix of size $(N-1) \times (N-1)$. Let $I_{(N-1) \times (N-1)}$ be a modification of $I_{N \times N}$ such that every column after the $(N-i)$th column is zero. Then create a new matrix, $W^T$, as:

\[
W^T = \begin{bmatrix} W^{(c)^T} & 0 \\ 0 & W^{(s)^T} \end{bmatrix}
\]
We obtain the following constrained least squares system:

\[
\begin{bmatrix}
K^T K & W \\
W^T & 0
\end{bmatrix}
\begin{bmatrix}
\Sigma \\
\Gamma
\end{bmatrix} =
\begin{bmatrix}
K^T S_{2N \times 2N} \\
1
\end{bmatrix}.
\]

The first part, \( K \Sigma = S_{2N \times 2N} \), accounts for the equation, \( \int_{\Omega} \sigma \bar{v} u \cdot v d\mathbf{x} = S_{2N \times 2N} \) in \( \Omega \).

The second part, \( W^T \Sigma = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \), incorporates the boundary conditions enforced upon \( \bar{\sigma} \) and \( \sigma_R \).
CHAPTER 5: Demonstrations of the Method for recovering $\sigma_R(r, \theta)$

The ability to recover $\sigma_J$ both depends on how detailed the boundary data is and on the selection of the sub-space used in expanding $\sigma_R$. Notice that the space for the boundary data limits the dimension of $\Pi_q = \text{span}\{\mathcal{V}E(1)\phi_k \cdot \mathcal{V}E(1)\psi_k | \phi_k, \psi_k \in V_{2q}\}$. This in turn limits selection of the sub-space used in expanding $\sigma_R$ if one wants to guarantee a unique solution. As had been previously demonstrated, when the number of basis functions used to describe $g$ is truncated to dimension $q = N$. This truncated version is $g_N$. The dimension of $\Pi_q$ will also reflect this truncation as $q^2$. Seeking a restriction that guarantees a unique solution for $\sigma_R$ then gives $(q - 1)^2$ after $\sigma_R$’s boundary conditions are included as constraints. In order words, we pretend that the true $\sigma_J$ is $\sigma_R$ and that it lives in a sub-space of $\Pi_q$. However, if the complete expansion of the true $\sigma_J$ has components orthogonal to the selected sub-space, these components will not be represented in the expansion of $\sigma_R$. This problem will prevent the expansion of $\sigma_R$ from perfectly representing $\sigma_J$. Likewise, for the same reason, $\sigma_J$ might not necessarily be a good representation of $\sigma$. Another problem in recovering $\sigma_R$ is when the ranges for the powers and the frequencies used for the recovery are too large. (Recall the range used in the recovery is described as the range for $\mathcal{I}$ with $j \leq \mathcal{I} = 2(i - 1) + j \leq 2(N - j - 1) + j = 2N - j - 2$ and $0 \leq j \leq N - 1 = N_\sigma$. Also note that the basis set used to construct $\mathbf{K}$ in the recovery is not orthogonal). Since the basis set used is not orthogonal, we might have basis functions present in the recovered coefficient that are not present in the original $\sigma_J$. In addition, errors inserted when numerically obtaining $\sigma_J$ expansion and $u$ can contribute in reducing the quality of the recovered conductivity coefficient. The errors will then carry into the error obtained when calculating $S_{2N \times 2N}$. These can then cause $\sigma_R$ to defer from $\sigma_J$.

The first example is uses $\sigma = 1$ everywhere on the unit disk.
Figure 1  Actual $\sigma_{ji}$ with $I = J = 10$

Figure 2  Reconstructed $\sigma_R$ with $N_\sigma = N - 1 = 7$. 
Figure 3  Boundary values of recovered $\sigma_R$ when $\sigma_{JI} = 1$ everywhere on the unit disk.

We will now try to recover two types of expansions of the following:

$$\sigma = (1 - r) \left( r^2 + r\cos(\theta) + (r\sin(\theta))^2 + 1 \right) + r.$$
Figure 4  Actual $\sigma_H$ with $I = J = 10$
Figure 5  Reconstructed $\sigma_R$ with $N = M = 11$, $N_\rho = N - 6 = 5$
Figure 6  
Boundary values of recovered $\sigma_R$.  

Reconstructed Conductivity Evaluated on the boundary
Figure 7  Reconstructed $\sigma_R$ with $N = M = 11$, $N_\sigma = N - 4 = 7$ too many extra basis functions were included in this case.

We will now increase the maximum frequency by replacing $(r\sin(\theta))$ with $3(r\sin(\theta))^8$ and try to recover two types of expansions of the following new expression:

$$\sigma = (1 - r) \left( r^2 + r\cos(\theta) + 3(r\sin(\theta))^8 + 1 \right) + r.$$
Figure 8  Actual $\sigma_H$ with $I = J = 10$
Reconstructed $\sigma_h$ with $M = 11$, $N_e = N - 5 = 6$. Too many extra basis functions were included in this case and the error inserted when calculating $S_{2N \times 2N}$ may have been large.

We then increase the coefficient, 3, in $3(r \sin(\theta))^8$ to 13:

$$\sigma = (1 - r)\left( r^2 + r\cos(\theta) + 13(r\sin(\theta))^9 + 1 \right) + r.$$
Figure 10  Actual $\sigma_{\mu}$ when $I = J = 10$
Reconstructed $\sigma_R$ with $N = M = 11$, $N_\sigma = N - 5 = 6$. Too many extra basis functions were included in this case and the error inserted when calculating $S_{2N \times 2N}$ may have been large.

The extra bumps illustrate the great need to use orthogonal basis functions when constructing the least-squares matrix, $K$, and when calculating $U$ and $S_{2N \times 2N}$. 
APPENDIX A: The Gradient in polar coordinates

Let
\[ \begin{align*}
  f &= f(x, y) \\
  x &= r \cos \theta \\
  y &= r \sin \theta
\end{align*} \]

\[ \nabla f = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial \theta} + \frac{\partial f}{\partial r} r \]

Then:
\[ \begin{align*}
  \frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x} (-r \sin \theta) + \frac{\partial f}{\partial y} (r \cos \theta) \\
  \frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x} (\cos \theta) + \frac{\partial f}{\partial y} (\sin \theta)
\end{align*} \]

This means that the gradient formula in polar coordinates can be represented as:
\[ \begin{align*}
  \left[ \begin{array}{c}
  \frac{\partial f}{\partial x} \\
  \frac{\partial f}{\partial y} \\
  \frac{\partial f}{\partial \theta}
  \end{array} \right] &= \frac{1}{r} \left[ \begin{array}{ccc}
  \sin \theta & -r \cos \theta \\
  -\cos \theta & -r \sin \theta \\
  0 & 1
  \end{array} \right] \left[ \begin{array}{c}
  \frac{\partial f}{\partial x} \\
  \frac{\partial f}{\partial y} \\
  \frac{\partial f}{\partial \theta}
  \end{array} \right]
\]
References

1 See Electrical Impedance Tomography, by Liliana Borcea


3 See Electrical Impedance Tomography, by Liliana Borcea
   http://www.msri.org/communications/ln/msri/2001/jiw2001/borcea/1/banner/14.html (There, \( \gamma = \sigma + i\omega \). We only explore \( \sigma \) in our report).

