

# TOPOLOGICAL QUANTUM INFORMATION AND THE JONES POLYNOMIAL

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Arxiv: 1001.0354

# Background Ideas and References

# Topological Quantum Information Theory (with Sam Lomonaco)

quant-ph/0603131 and quant-ph/0606114

## Spin Networks and Anyonic TQC

[arXiv:0805.0339](#) **Quantum Knots and Mosaics**

[arXiv:0910.5891](#)

[arXiv:0804.4304](#) The Fibonacci Model and the Temperley-Lieb Algebra

[arXiv:0706.0020](#) A 3-Stranded Quantum Algorithm for the Jones Polynomial

[arXiv:0909.1080](#) NMR Quantum Calculations of the Jones Polynomial  
**Authors:** Raimund [Marx](#), Amr [Fahmy](#), Louis [Kauffman](#),

Samuel [Lomonaco](#), Andreas [Spörl](#), Nikolas [Pomplun](#), John [Myers](#), Steffen J. [Glaser](#)

[arXiv:0909.1672](#) Anyonic topological quantum computation and the virtual braid group. H. Dye and LK.

# Untying Knots by NMR: first experimental implementation of a quantum algorithm for approximating the Jones polynomial

Raimund Marx<sup>1</sup>, Andreas Spörl<sup>1</sup>, Amr F. Fahmy<sup>2</sup>, John M. Myers<sup>3</sup>, Louis H. Kauffman<sup>4</sup>, Samuel J. Lomonaco, Jr.<sup>5</sup>, Thomas-Schulte-Herbrüggen<sup>1</sup>, and Steffen J. Glaser<sup>1</sup>

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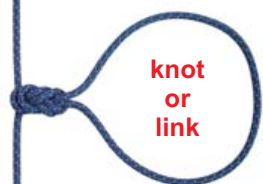
<sup>2</sup>Harvard Medical School, 25 Shattuck Street, Boston, MA 02115, U.S.A.

<sup>3</sup>Gordon McKay Laboratory, Harvard University, 29 Oxford Street, Cambridge, MA 02138, U.S.A.

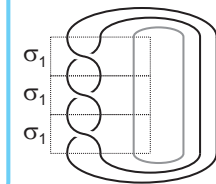
<sup>4</sup>University of Illinois at Chicago, 851 S. Morgan Street, Chicago, IL 60607-7045, U.S.A.

<sup>5</sup>University of Maryland Baltimore County, 1000 Hilltop Circle, Baltimore, MD 21250, U.S.A.

## roadmap of the quantum algorithm



### example #1 Trefoil



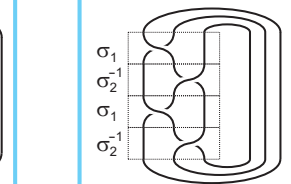
$$U_{\text{Trefoil}} = (U_1)^3$$

$$U_1 = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & -e^{-i\theta} \frac{\sin(4\theta)}{\sin(2\theta)} + e^{-i\theta} \end{pmatrix}$$

Step #1: from the 2x2 matrix  $U$  to the 4x4 matrix  $cU$ :

$$cU = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$$

### example #2 Figure-Eight



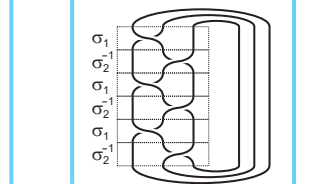
$$U_{\text{Figure-Eight}} = (U_2^{-1} \cdot U_1)^2$$

$$U_2 = \begin{pmatrix} -e^{-i\theta} \frac{\sin(6\theta)}{\sin(4\theta)} + e^{-i\theta} & -e^{-i\theta} \frac{\sqrt{\sin(6\theta)\sin(2\theta)}}{\sin(4\theta)} \\ -e^{-i\theta} \frac{\sqrt{\sin(6\theta)\sin(2\theta)}}{\sin(4\theta)} & -e^{-i\theta} \frac{\sin(2\theta)}{\sin(4\theta)} + e^{-i\theta} \end{pmatrix}$$

Step #2: application of  $cU$  on the NMR product operator  $I_{1x}$ :

$$cU I_{1x} cU^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}^\dagger = \frac{1}{2} \begin{pmatrix} 0 & U^\dagger \\ U & 0 \end{pmatrix}$$

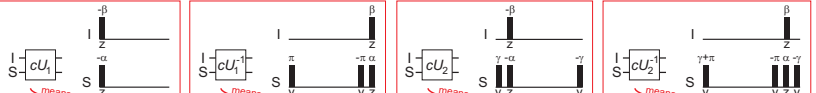
### example #3 Borromean rings



$$U_{\text{Borrom.R.}} = (U_2^{-1} \cdot U_1)^3$$

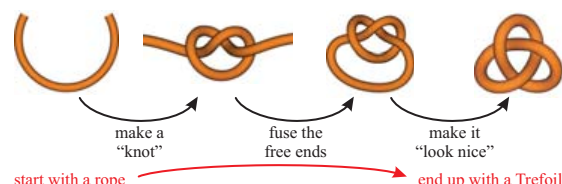
Step #3: measurement of  $I_{1x}$  and  $I_{1y}$ :

$$\begin{aligned} \text{tr} \left\{ I_{1x} \frac{1}{2} \begin{pmatrix} 0 & U^\dagger \\ U & 0 \end{pmatrix} \right\} &= \frac{1}{2} \Re(\text{tr}(U)) \\ \text{tr} \left\{ I_{1y} \frac{1}{2} \begin{pmatrix} 0 & U^\dagger \\ U & 0 \end{pmatrix} \right\} &= \frac{1}{2} \Im(\text{tr}(U)) \end{aligned}$$



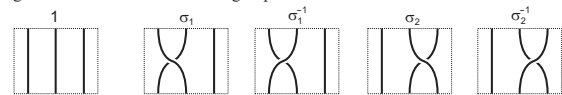
A knot is defined as a closed, non-self-intersecting curve that is embedded in three dimensions.

example: "construction" of the Trefoil knot:



J. W. Alexander proved, that any knot can be represented as a closed braid (polynomial time algorithm)

generators of the 3 strand braid group:



It is well known in knot theory, how to obtain the unitary matrix representation of all generators of a given braid group (see "Temperley-Lieb algebra" and "path model representation"). The unitary matrices  $U_1$  and  $U_2$ , corresponding to the generators  $\sigma_1$  and  $\sigma_2$  of the 3 strand braid group are shown on the left, where the variable " $\theta$ " is related to the variable " $A$ " of the Jones polynomial by:  $A = e^{-i\theta}$ . The unitary matrix representations of  $\sigma_1^{-1}$  and  $\sigma_2^{-1}$  are given by  $U_1^{-1}$  and  $U_2^{-1}$ .

The knot or link that was expressed as a product of braid group generators can therefore also be expressed as a product of the corresponding unitary matrices.

Instead of applying the unitary matrix  $U$ , we apply its controlled variant  $cU$ . This matrix is especially suited for NMR quantum computers [4] and other thermal state expectation value quantum computers: you only have to apply  $cU$  to the NMR product operator  $I_{1x}$  and measure  $I_{1x}$  and  $I_{1y}$  in order to obtain the trace of the original matrix  $U$ .

Independent of the dimension of matrix  $U$  you only need ONE extra qubit for the implementation of  $cU$  as compared to the implementation of  $U$  itself.

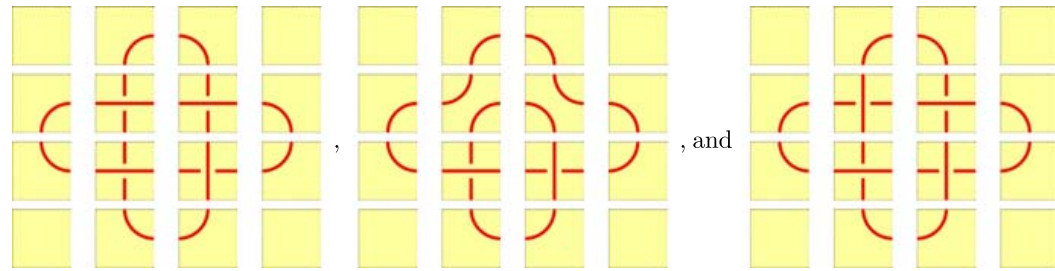
The measurement of  $I_{1x}$  and  $I_{1y}$  can be accomplished in one single-scan experiment.

All knots and links can be expressed as a product of braid group generators (see above). Hence the corresponding NMR pulse sequence can also be expressed as a sequence of NMR pulse sequence blocks, where each block corresponds to the controlled unitary matrix  $cU$  of one braid group generator.



# Quantum knots and mosaics

with  
Sam  
Lomonaco



Each of these knot mosaics is a string made up of the following 11 symbols



called *mosaic tiles*.

Each mosaic is a tensor product of  
elementary tiles.

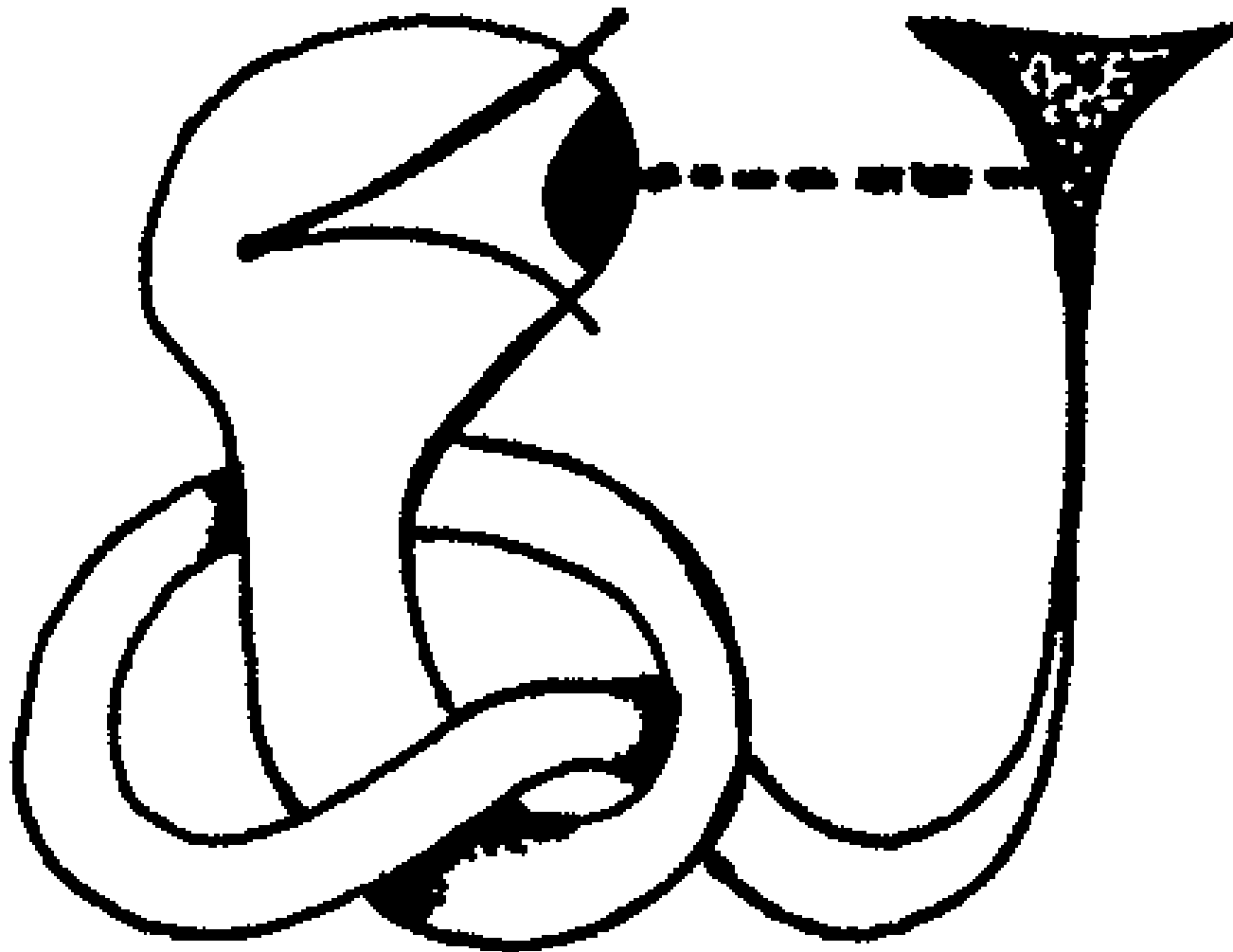
$$\Omega = \left| \begin{array}{cccc} \text{tile} & \text{tile} & \text{tile} & \text{tile} \\ \text{tile} & \text{tile} & \text{tile} & \text{tile} \\ \text{tile} & \text{tile} & \text{tile} & \text{tile} \\ \text{tile} & \text{tile} & \text{tile} & \text{tile} \end{array} \right\rangle \left\langle \begin{array}{cccc} \text{tile} & \text{tile} & \text{tile} & \text{tile} \\ \text{tile} & \text{tile} & \text{tile} & \text{tile} \\ \text{tile} & \text{tile} & \text{tile} & \text{tile} \\ \text{tile} & \text{tile} & \text{tile} & \text{tile} \end{array} \right| + \left| \begin{array}{cccc} \text{tile} & \text{tile} & \text{tile} & \text{tile} \\ \text{tile} & \text{tile} & \text{tile} & \text{tile} \\ \text{tile} & \text{tile} & \text{tile} & \text{tile} \\ \text{tile} & \text{tile} & \text{tile} & \text{tile} \end{array} \right\rangle \left\langle \begin{array}{cccc} \text{tile} & \text{tile} & \text{tile} & \text{tile} \\ \text{tile} & \text{tile} & \text{tile} & \text{tile} \\ \text{tile} & \text{tile} & \text{tile} & \text{tile} \\ \text{tile} & \text{tile} & \text{tile} & \text{tile} \end{array} \right|$$

This observable is a quantum knot invariant for 4x4 tile space. Knots have characteristic invariants in nxn tile space.

Superpositions of combinatorial knot configurations are seen as quantum states.

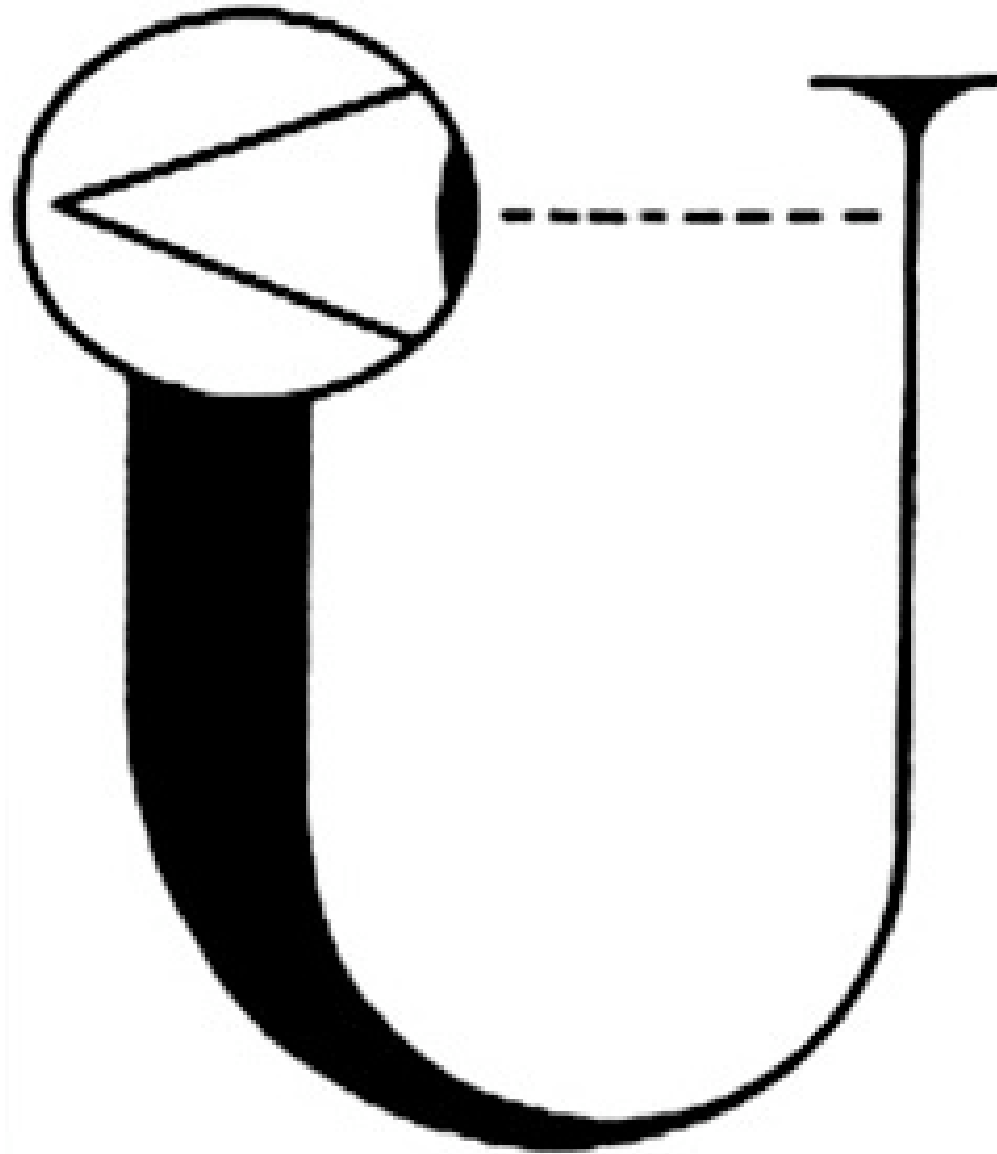
# The Grand Generalization

Universe as a Quantum Knot:  
Self-Excited Circuit Producing its Own Context





# The Wheeler Eye



# Papers on Quantum Computing, Knots, Khovanov Homology and Virtual Knot Theory

arXiv:1001.0354

**Title:** Topological Quantum Information, Khovanov Homology and the Jones Polynomial

**Authors:** Louis H. [Kauffman](#)

arXiv:0907.3178

**Title:** Remarks on Khovanov Homology and the Potts Model

**Authors:** Louis H. [Kauffman](#)

arXiv:0173783

**Title:** Introduction to virtual knot theory

**Author:** Louis H. Kauffman

# Quantum Mechanics in a Nutshell

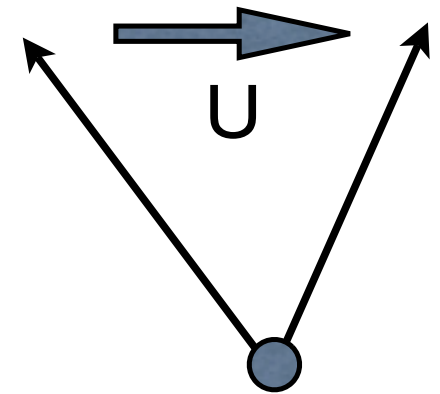
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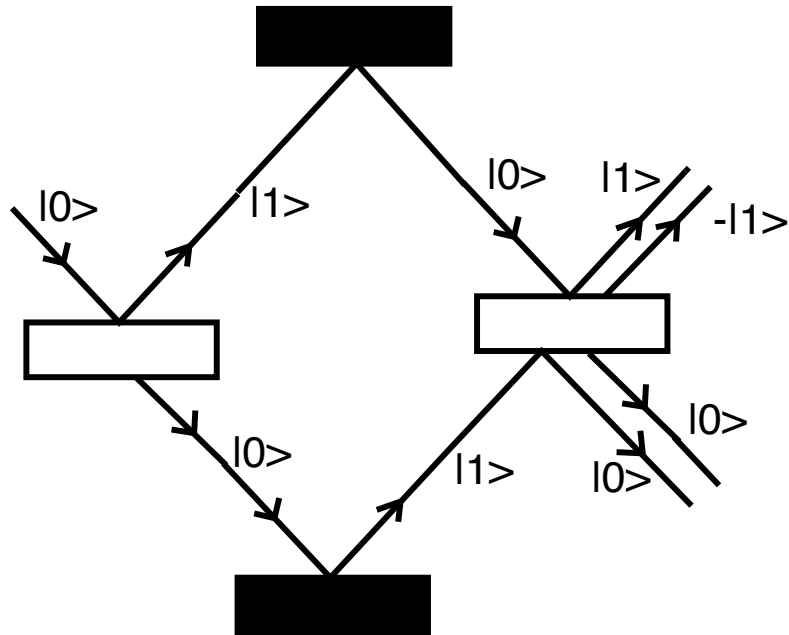
0. A state of a physical system corresponds to a unit vector  $|S\rangle$  in a complex vector space.

1. (measurement free) Physical processes are modeled by unitary transformations applied to the state vector:  $|S\rangle \longrightarrow U|S\rangle$

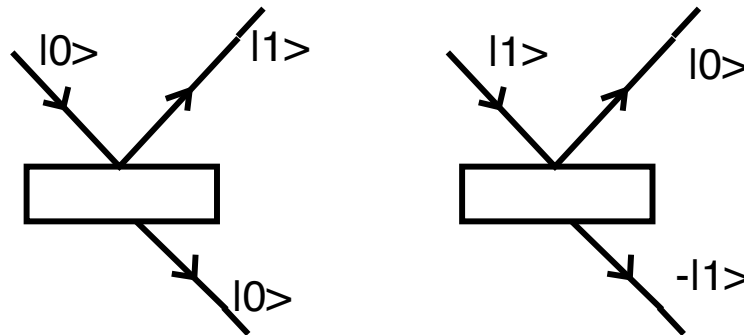
2. If  $|S\rangle = z_1|1\rangle + z_2|2\rangle + \dots + z_n|n\rangle$

in a measurement basis  $\{|1\rangle, |2\rangle, \dots, |n\rangle\}$ , then measurement of  $|S\rangle$  yields  $|i\rangle$  with probability  $|z_i|^2$ .





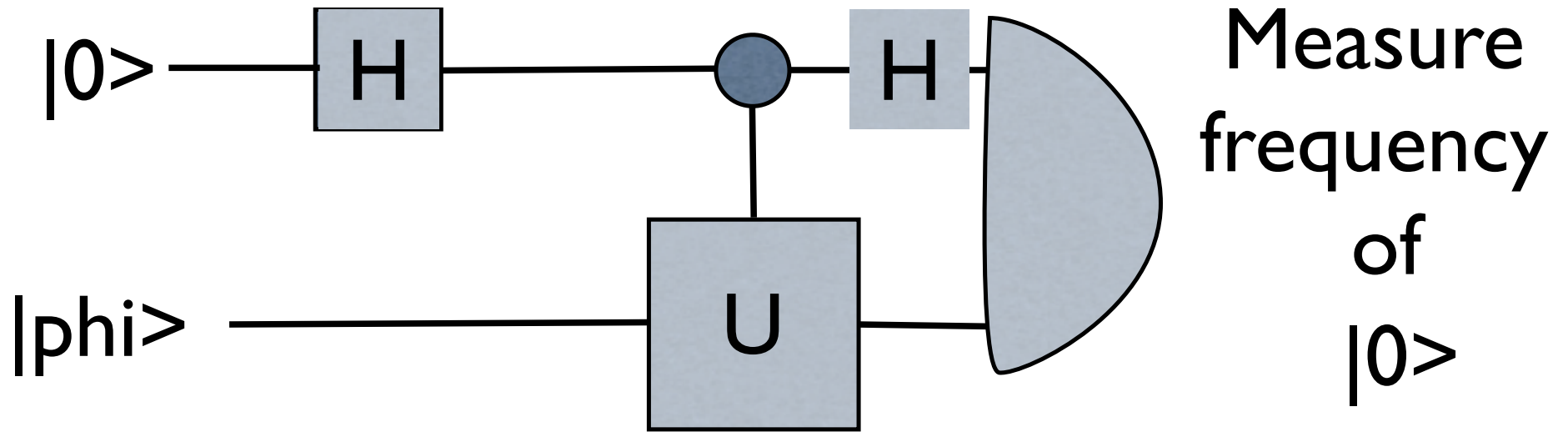
Mach-Zender Interferometer



$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$HMH = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

# Hadamard Test



$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$|0\rangle$  occurs with probability  $\frac{1}{2} + \text{Re}[\langle\phi|U|\phi\rangle]/2$ .

## Apply Hadamard Gate

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

to first qubit of

$$C_U \circ (H \otimes 1)|0\rangle|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes U|\psi\rangle)$$

**The resulting state is**

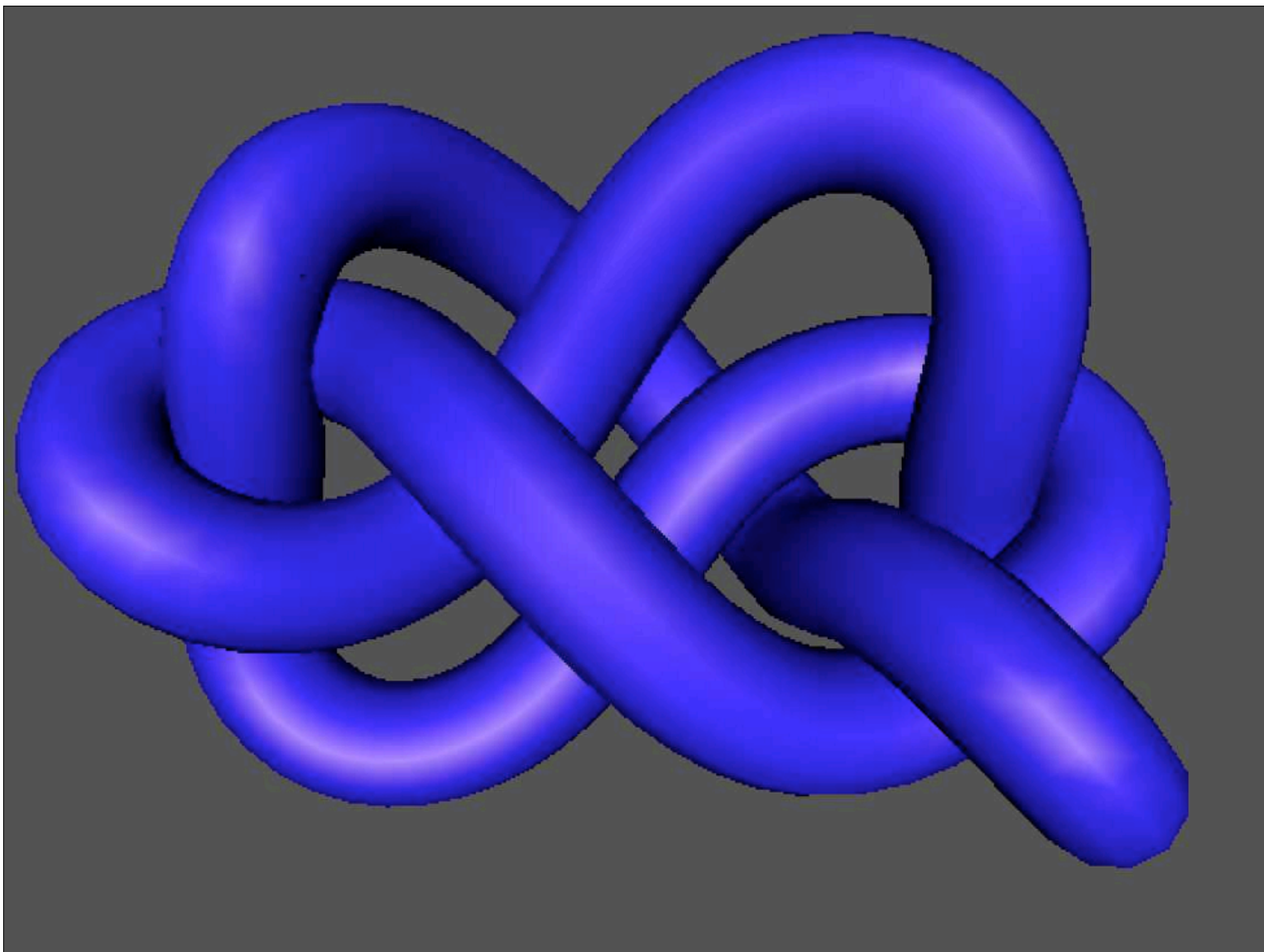
$$\begin{aligned} \frac{1}{2}(H|0\rangle \otimes |\psi\rangle + H|1\rangle \otimes U|\psi\rangle) &= \frac{1}{2}((|0\rangle + |1\rangle) \otimes |\psi\rangle + (|0\rangle - |1\rangle) \otimes U|\psi\rangle) \\ &= \frac{1}{2}(|0\rangle \otimes (|\psi\rangle + U|\psi\rangle) + |1\rangle \otimes (|\psi\rangle - U|\psi\rangle)). \end{aligned}$$

**The expectation for  $|0\rangle$  is**

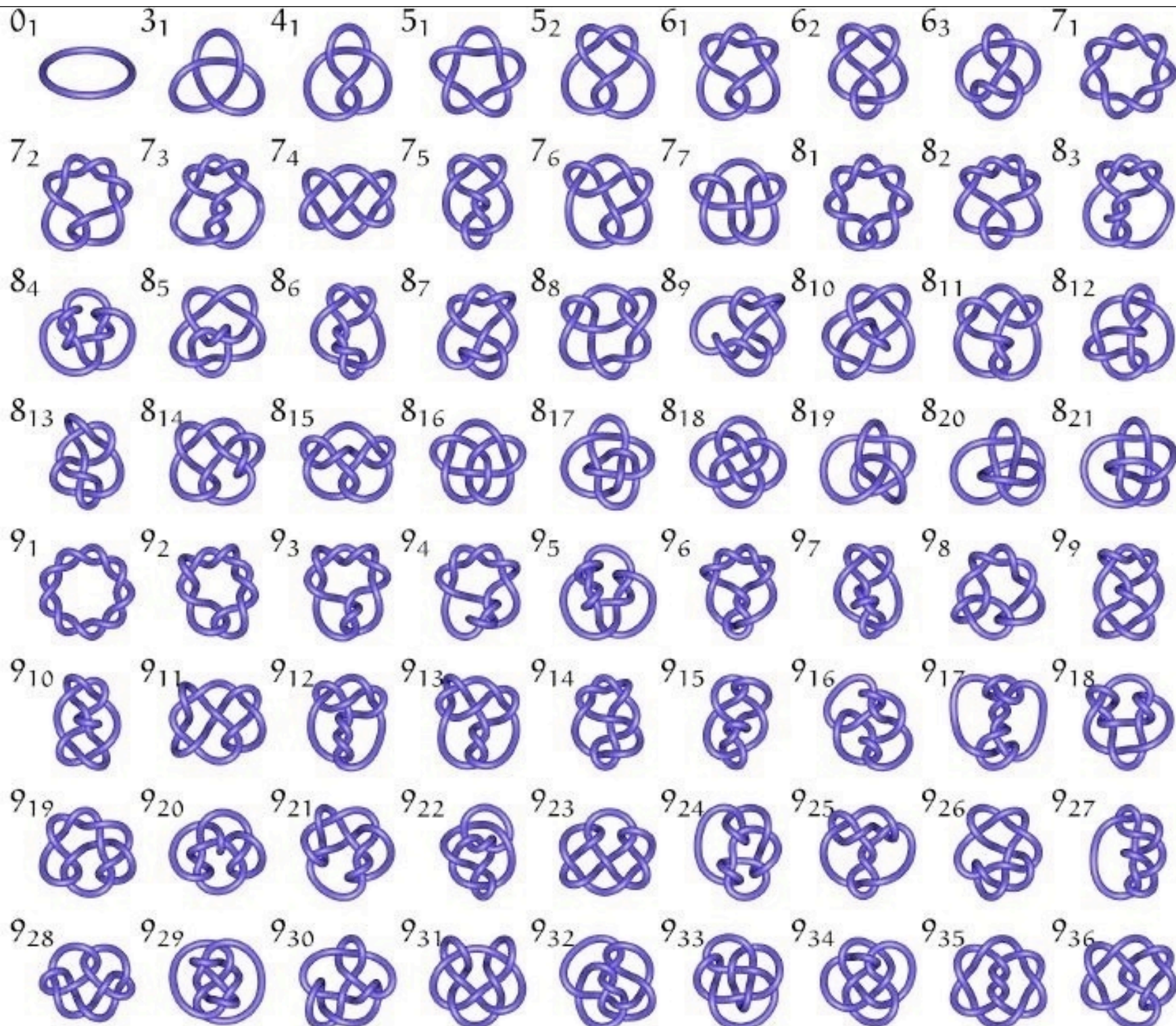
$$\frac{1}{2}(\langle\psi| + \langle\psi|U^\dagger)(|\psi\rangle + U|\psi\rangle) = \frac{1}{2} + \frac{1}{2}\text{Re}\langle\psi|U|\psi\rangle$$

**The imaginary part is obtained by applying the same procedure to**

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |\psi\rangle - i|1\rangle \otimes U|\psi\rangle)$$







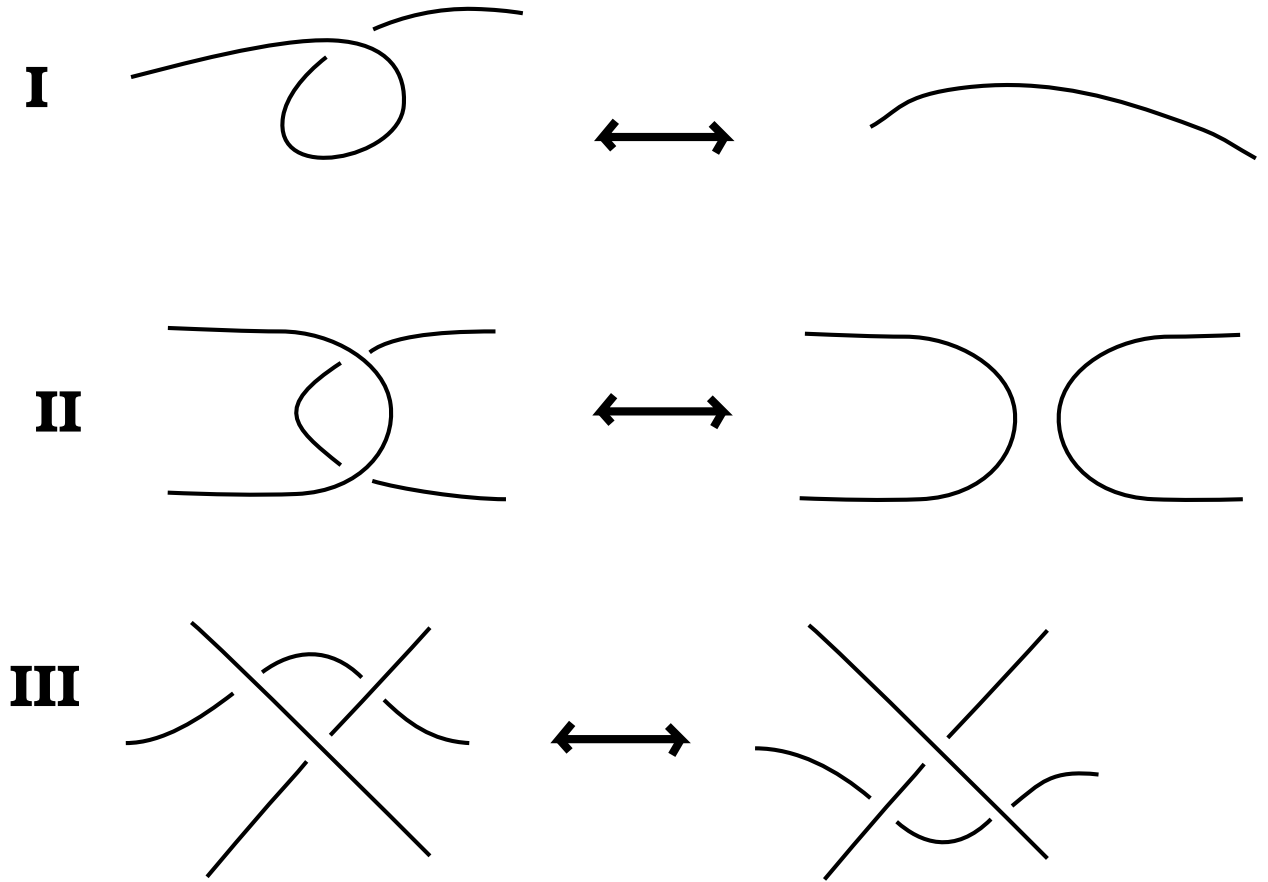
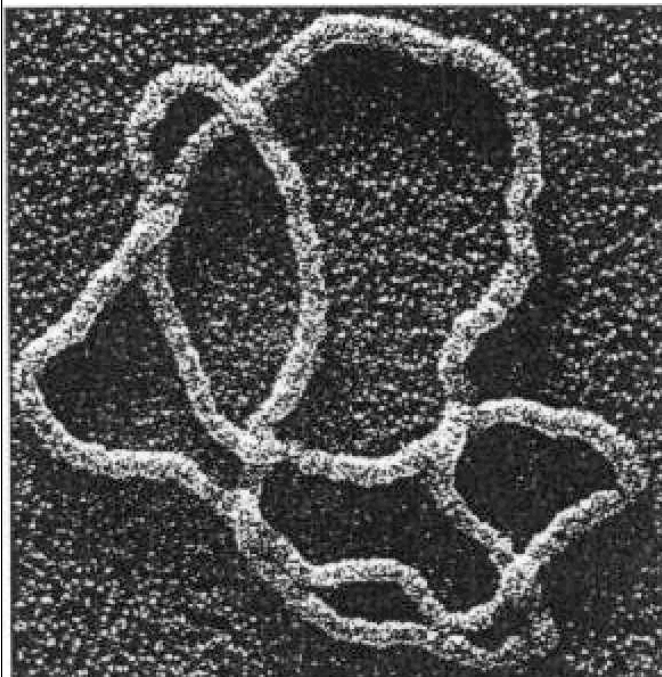
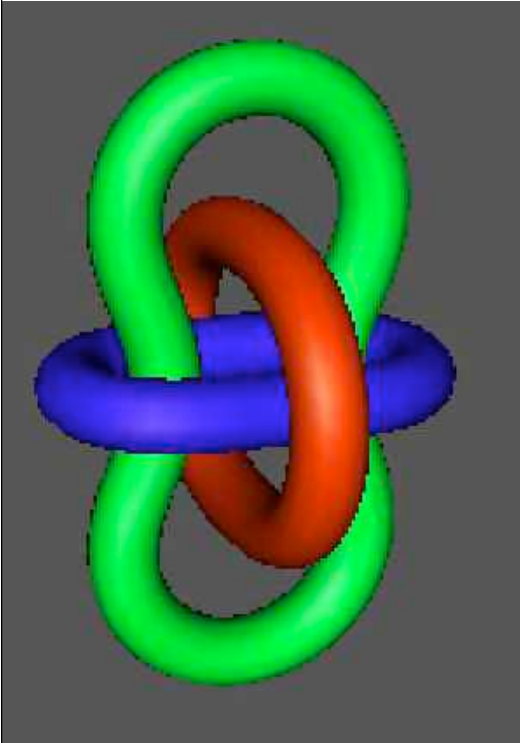


Figure 2 - The Reidemeister Moves.

Reidemeister Moves  
reformulate knot theory in  
terms of graph  
combinatorics.

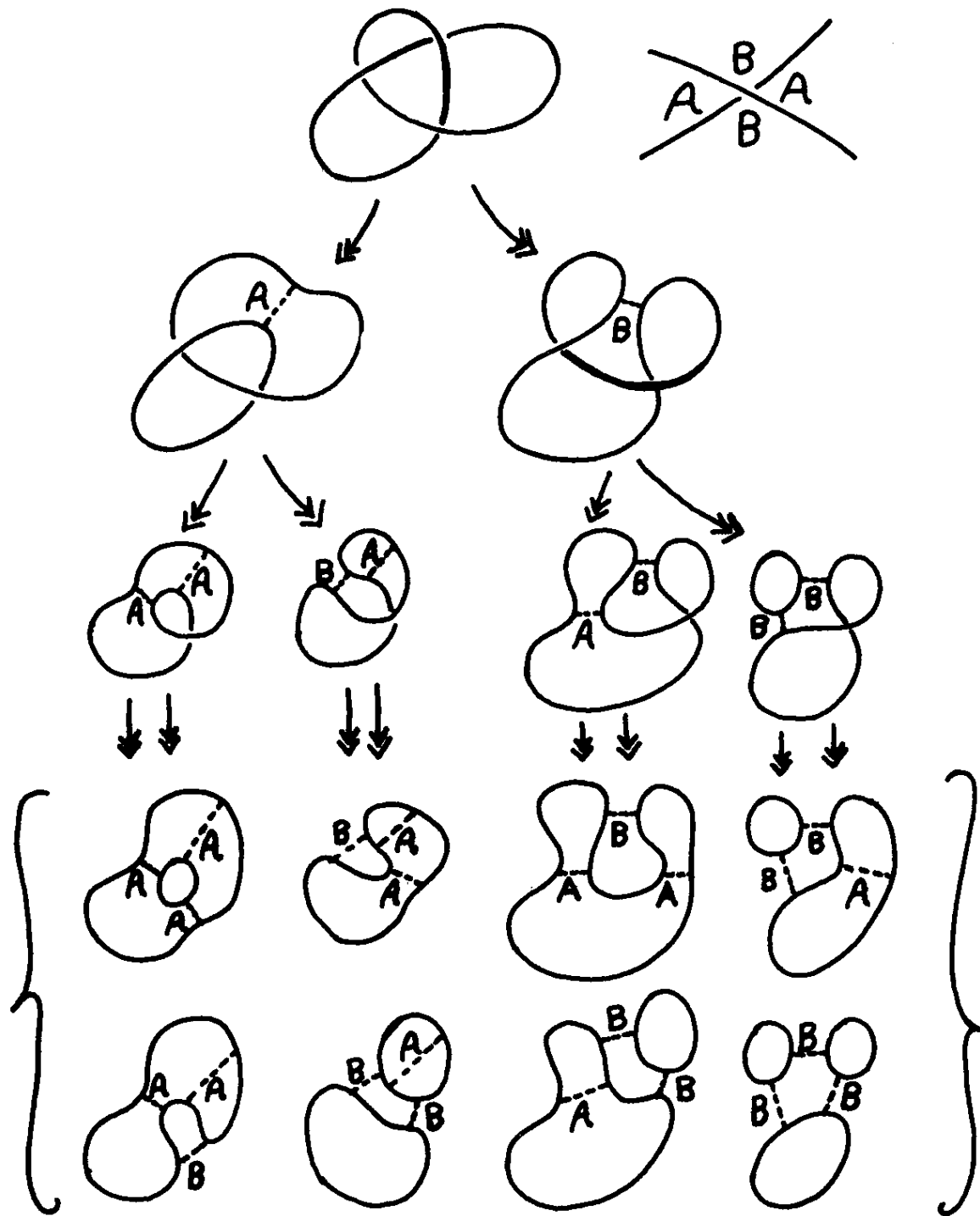
## Bracket Polynomial Model for the Jones Polynomial

$$\langle \text{crossing} \rangle = A \langle \text{positive crossing} \rangle + A^{-1} \langle \text{negative crossing} \rangle$$

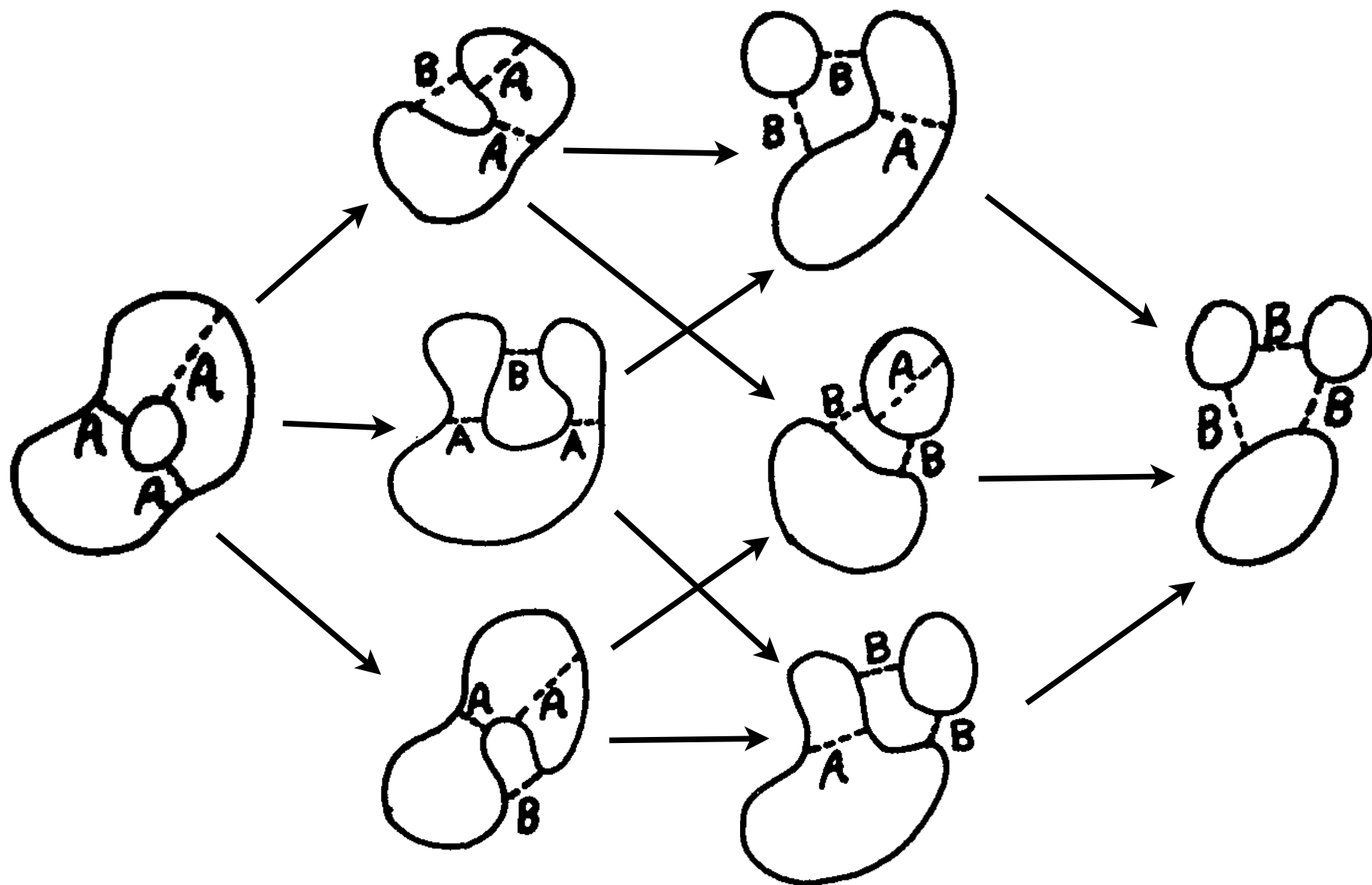
$$\langle K \circ \bigcirc \rangle = (-A^2 - A^{-2}) \langle K \rangle$$

$$\langle \text{curl} \rangle = (-A^3) \langle \text{cup} \rangle$$

$$\langle \text{uncurl} \rangle = (-A^{-3}) \langle \text{cup} \rangle$$



# The Khovanov Complex



## Reformulating the Bracket

Let  $c(K)$  = number of crossings on link  $K$ .

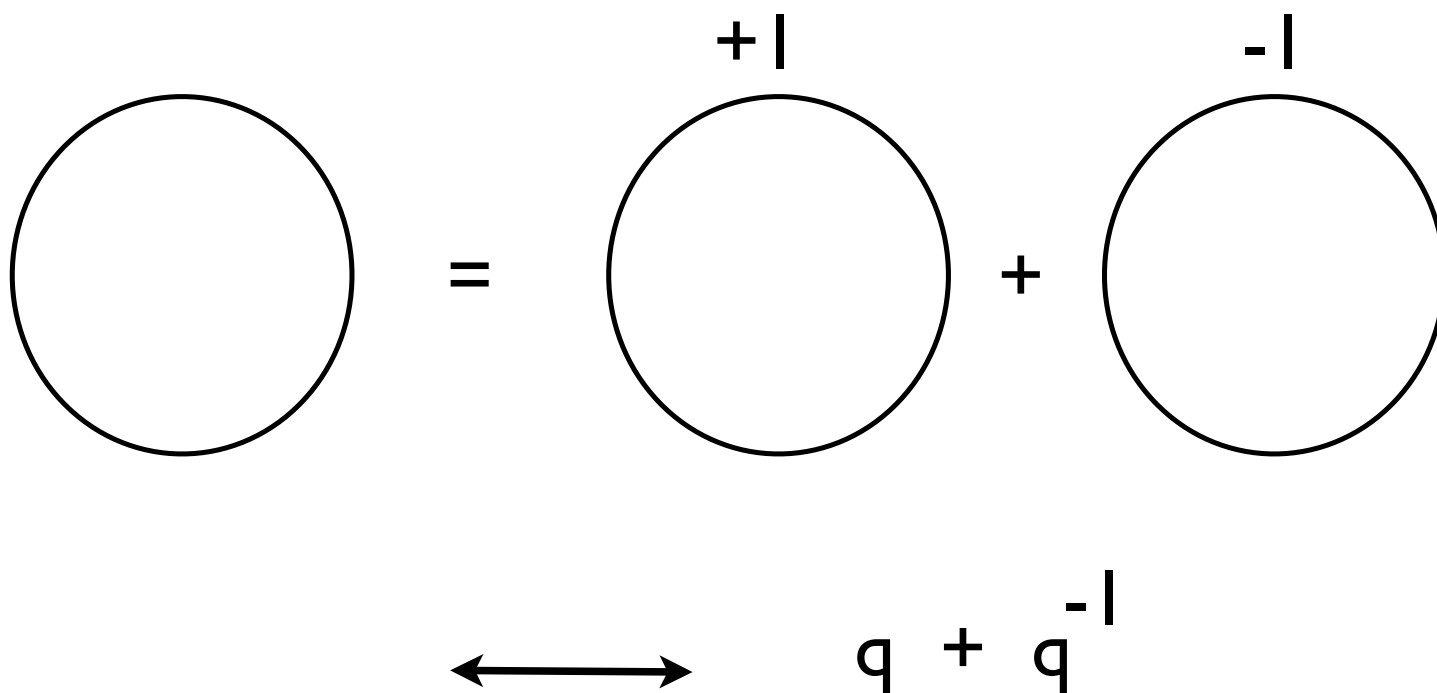
Form  $A^{-c(K)} \langle K \rangle$  and replace  $A^{-2}$  by  $-q$ .

Then the skein relation for  $\langle K \rangle$  will be replaced by:

$$\langle \text{crossing} \rangle = \langle \text{smoothing} \rangle - q \langle \text{empty} \rangle$$

$$\langle \bigcirc \rangle = (q + q^{-1})$$

Use enhanced states by labeling each loop with  
+1 or -1.



## Enhanced States

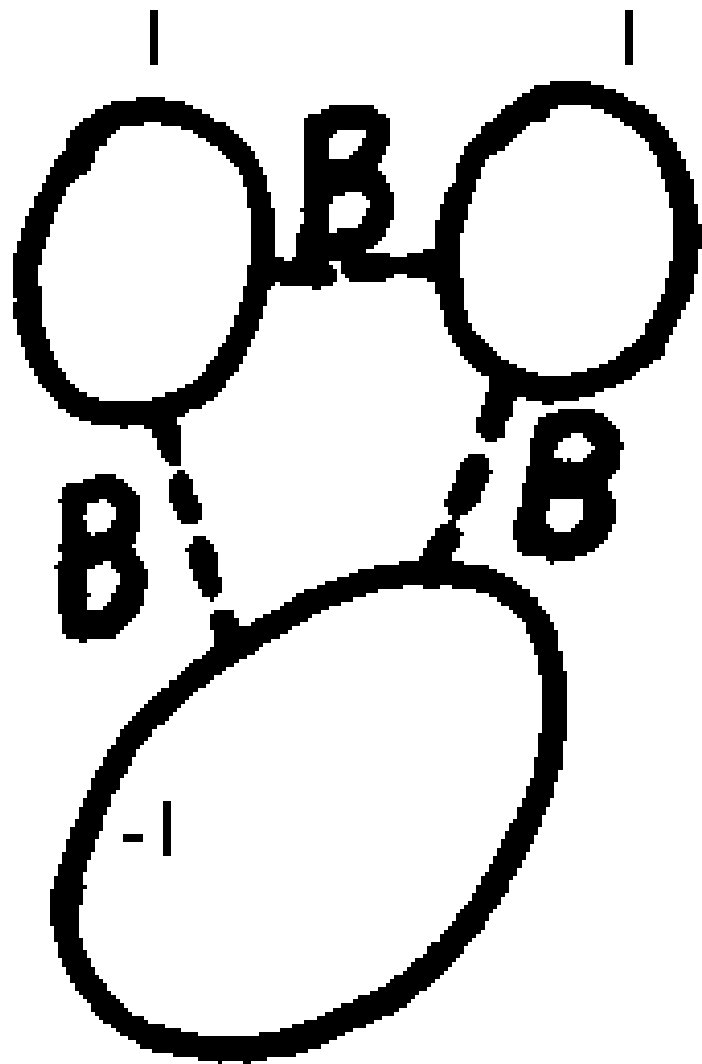
$$q^{-1} \iff -1 \iff X \circlearrowleft$$

$$q^{+1} \iff +1 \iff 1 \circlearrowleft$$

For reasons that will soon become apparent, we let  $-1$  be denoted by  $X$  and  $+1$  be denoted by  $1$ .

(The module  $V$  will be generated by  $1$  and  $X$ .)





An enhanced state  
that contributes

$$[(q)(q)(1/q)] [(-q) (-q) (-q)]$$

$$| \quad | \quad -| \quad \mathbf{B} \quad \mathbf{B} \quad \mathbf{B}$$

to the revised  
bracket state sum.

## Enhanced State Sum Formula for the Bracket

$$\langle K \rangle = \sum_s q^{j(s)} (-1)^{i(s)}$$

# A Quantum Statistical Model for the Bracket Polynomial.

Let  $\mathcal{C}(K)$  denote a Hilbert space with basis  $|s\rangle$  where  $s$  runs over the enhanced states of a knot or link diagram  $K$ .

We define a unitary transformation.

$$U : \mathcal{C}(K) \longrightarrow \mathcal{C}(K)$$

$$U|s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle$$

$q$  is chosen on the unit circle in the complex plane.

$$|\psi\rangle = \sum_s |s\rangle$$

**Lemma.** The evaluation of the bracket polynomial is given by the following formula

$$\langle K \rangle = \langle \psi | U | \psi \rangle.$$

This gives a new quantum algorithm for the Jones polynomial (via Hadamard Test).

# Generalization to Virtual Knot Theory

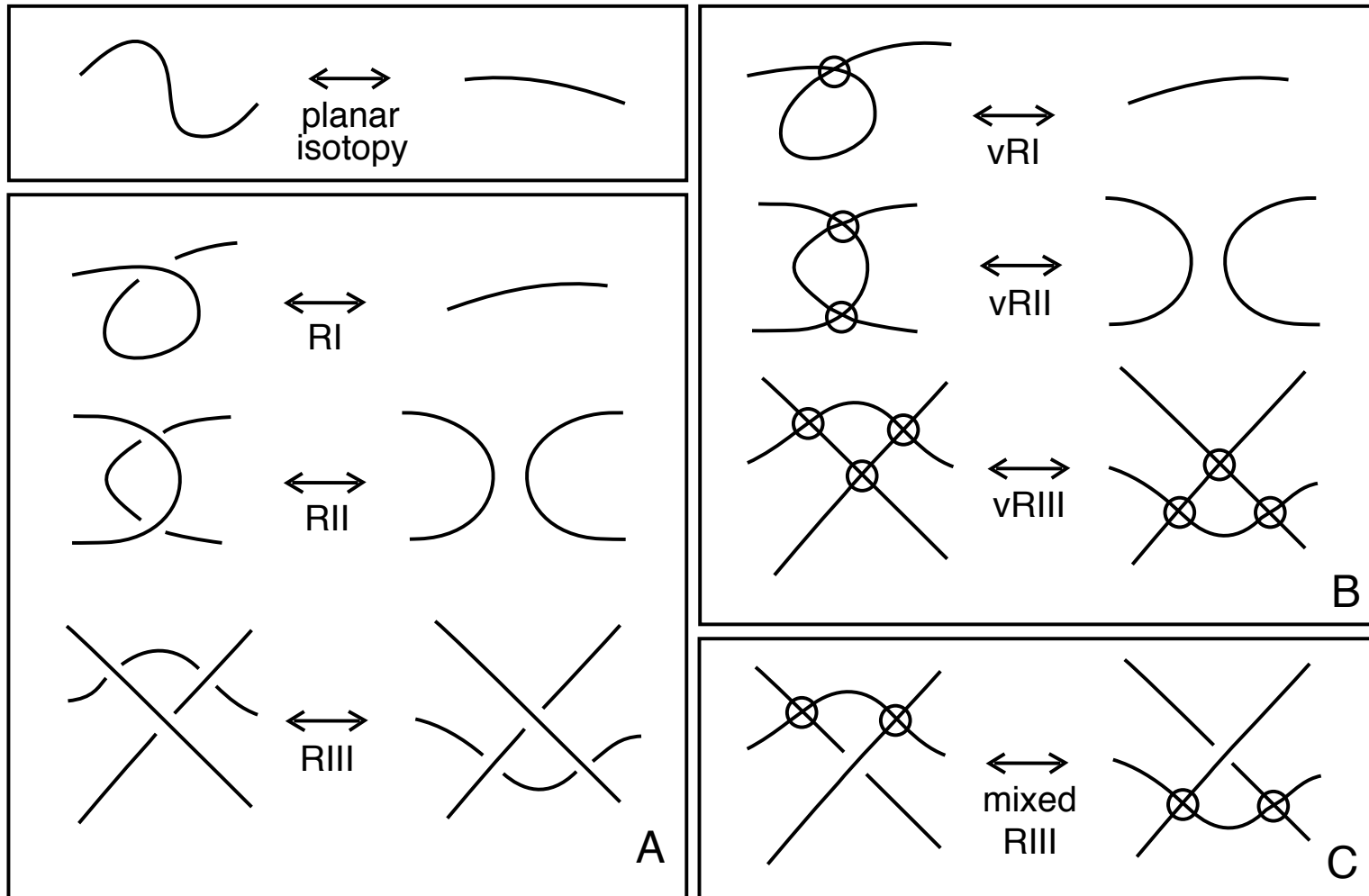


Figure 1: Moves

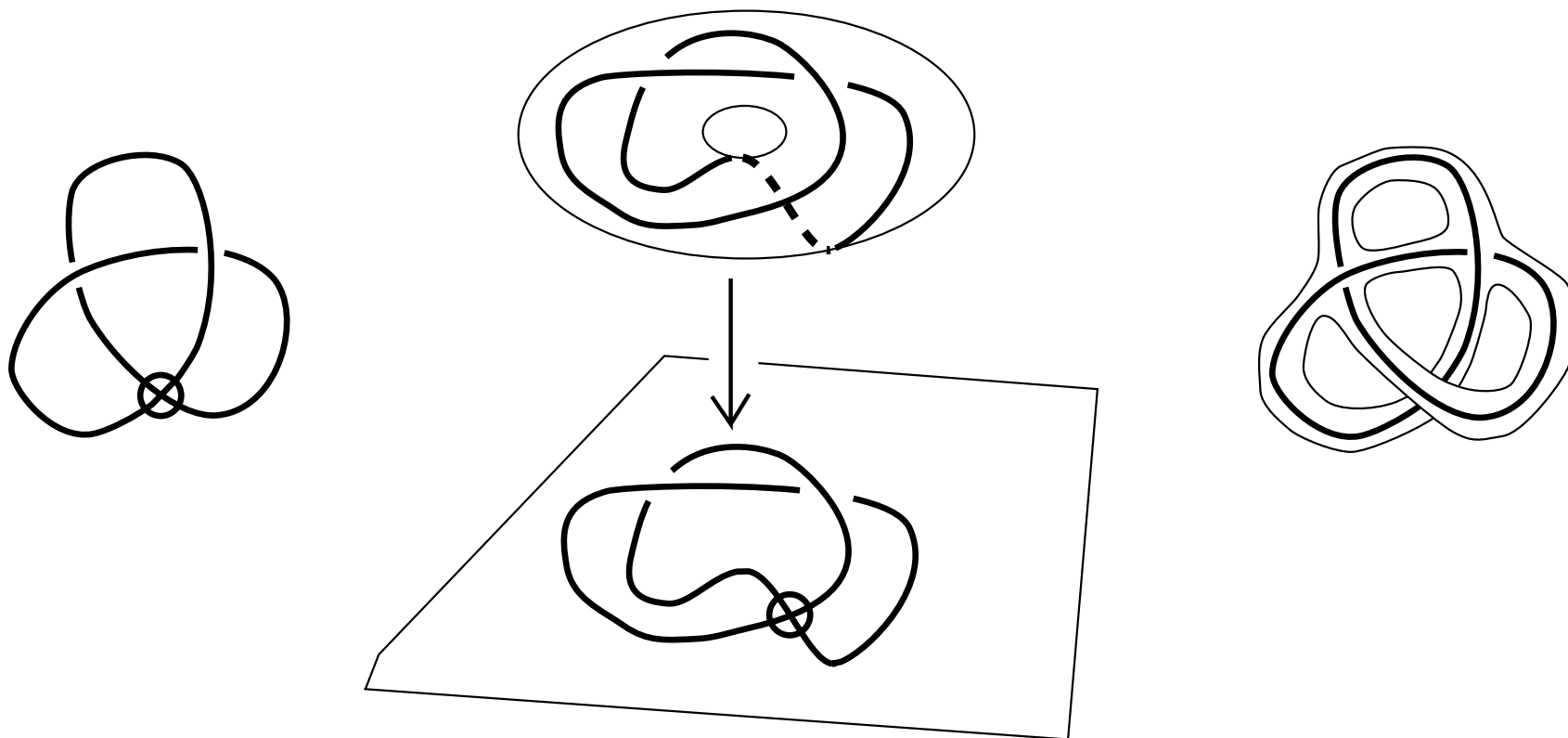


Figure 4: **Surfaces and Virtuals**

# A Quantum Algorithm for the Virtual Jones Polynomial

The bracket polynomial for virtual knots extends naturally by assigning the same values to extended states for loops that cross themselves virtually.

Thus the set of enhanced states of a virtual knot or link gives a basis for a Hilbert space, and therefore we have a quantum algorithm for the Jones polynomial extended to virtuals.

**AT THE PRESENT TIME THIS IS THE ONLY FORMULATION OF A QUANTUM ALGORITHM FOR THIS INVARIANT.**

# Khovanov Homology - Jones Polynomial as an Euler Characteristic

Two key motivating ideas are involved in finding the Khovanov invariant. First of all, one would like to *categorify* a link polynomial such as  $\langle K \rangle$ . There are many meanings to the term categorify, but here the quest is to find a way to express the link polynomial as a *graded Euler characteristic*  $\langle K \rangle = \chi_q \langle H(K) \rangle$  for some homology theory associated with  $\langle K \rangle$ .

**We will formulate Khovanov  
Homology  
in the context of our quantum  
statistical model for the bracket  
polynomial.**



# CATEGORIFICATION

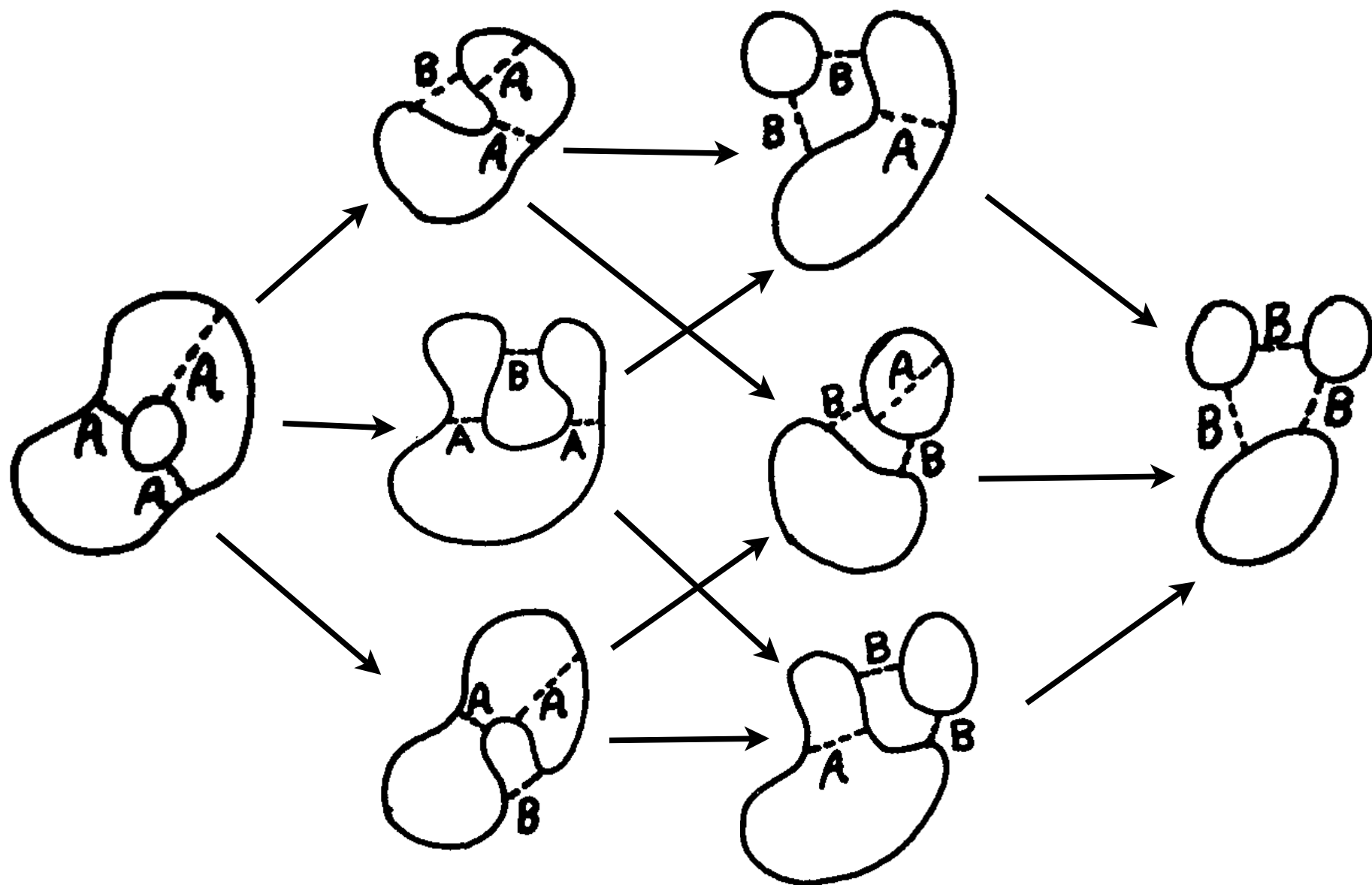
View the next slide of states of the bracket expansion as a CATEGORY.

The cubical shape of this category suggests making a homology theory.

In order to make a non-trivial homology theory we need a functor from this category of states to a module category. Each state loop will map to a module  $V$ . Unions of loops will map to tensor products of this module.

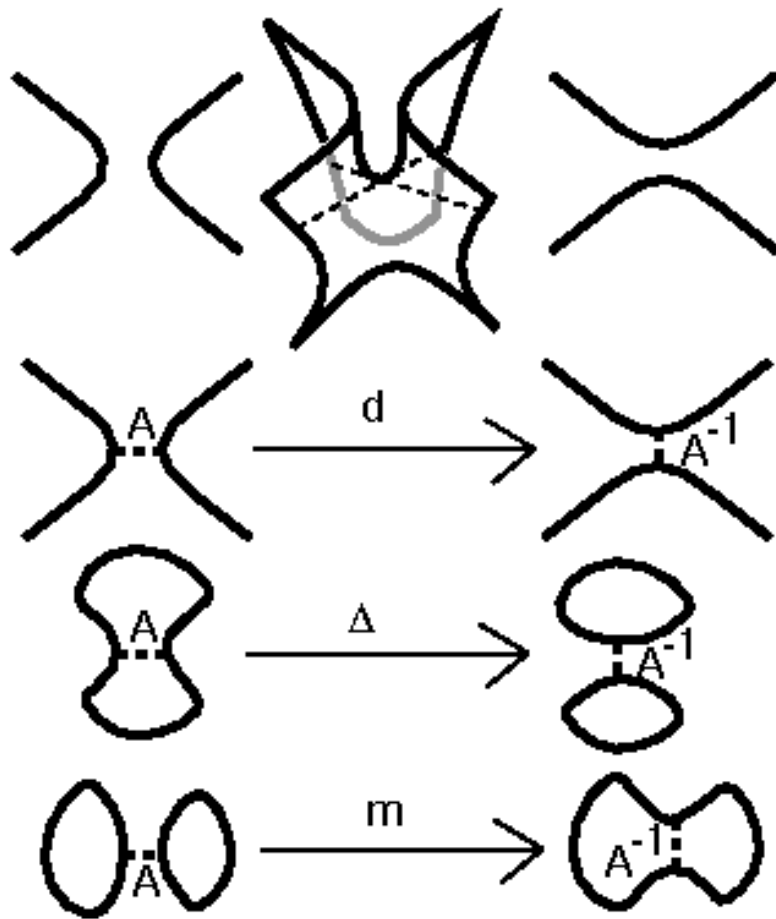
We will describe how this comes about after looking at the bracket polynomial in more detail.

# The Khovanov Complex



$$\partial(s) = \sum_{\tau} \partial_{\tau}(s)$$

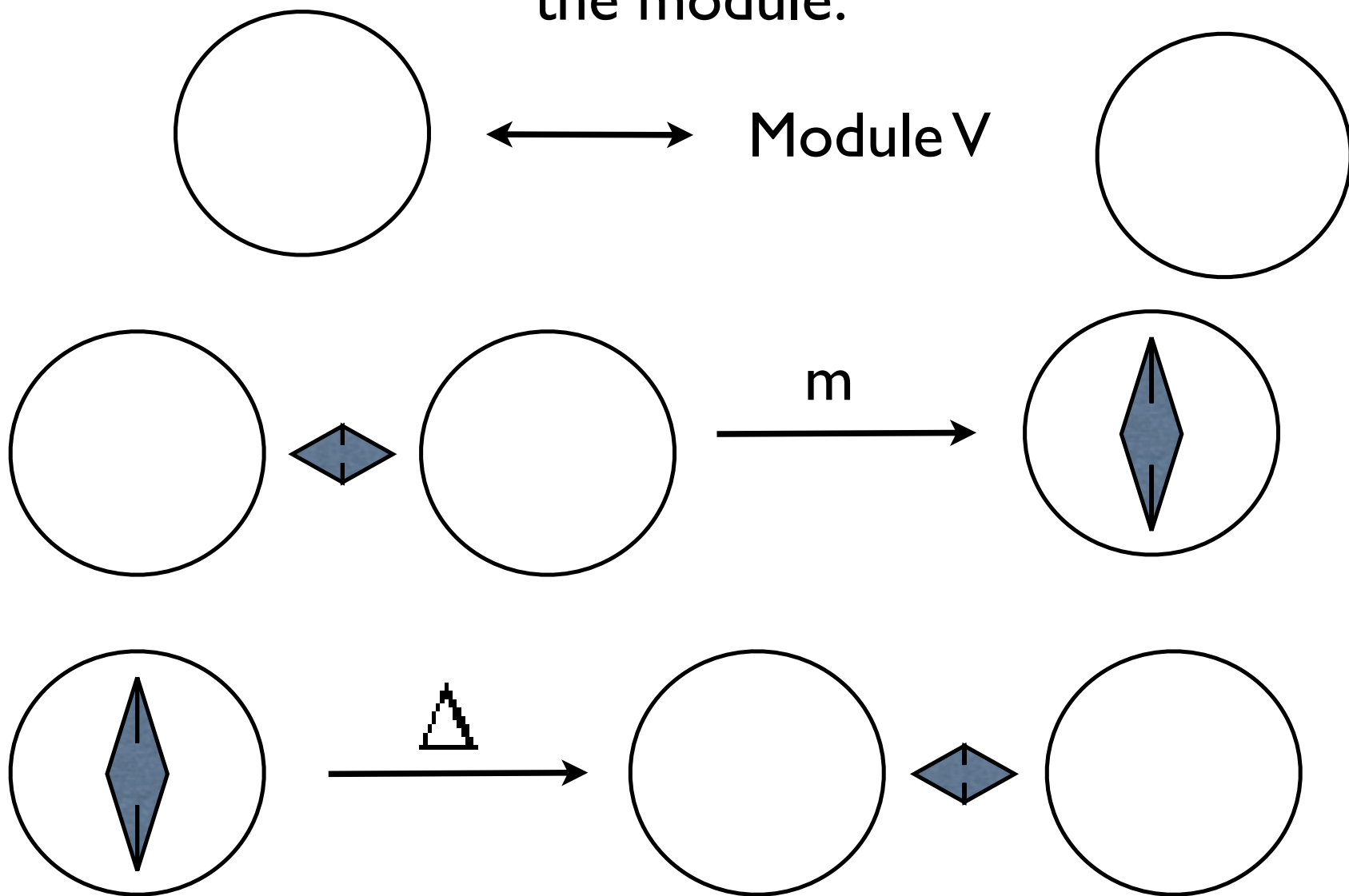
The boundary is a sum of partial differentials corresponding to resmoothings on the states.

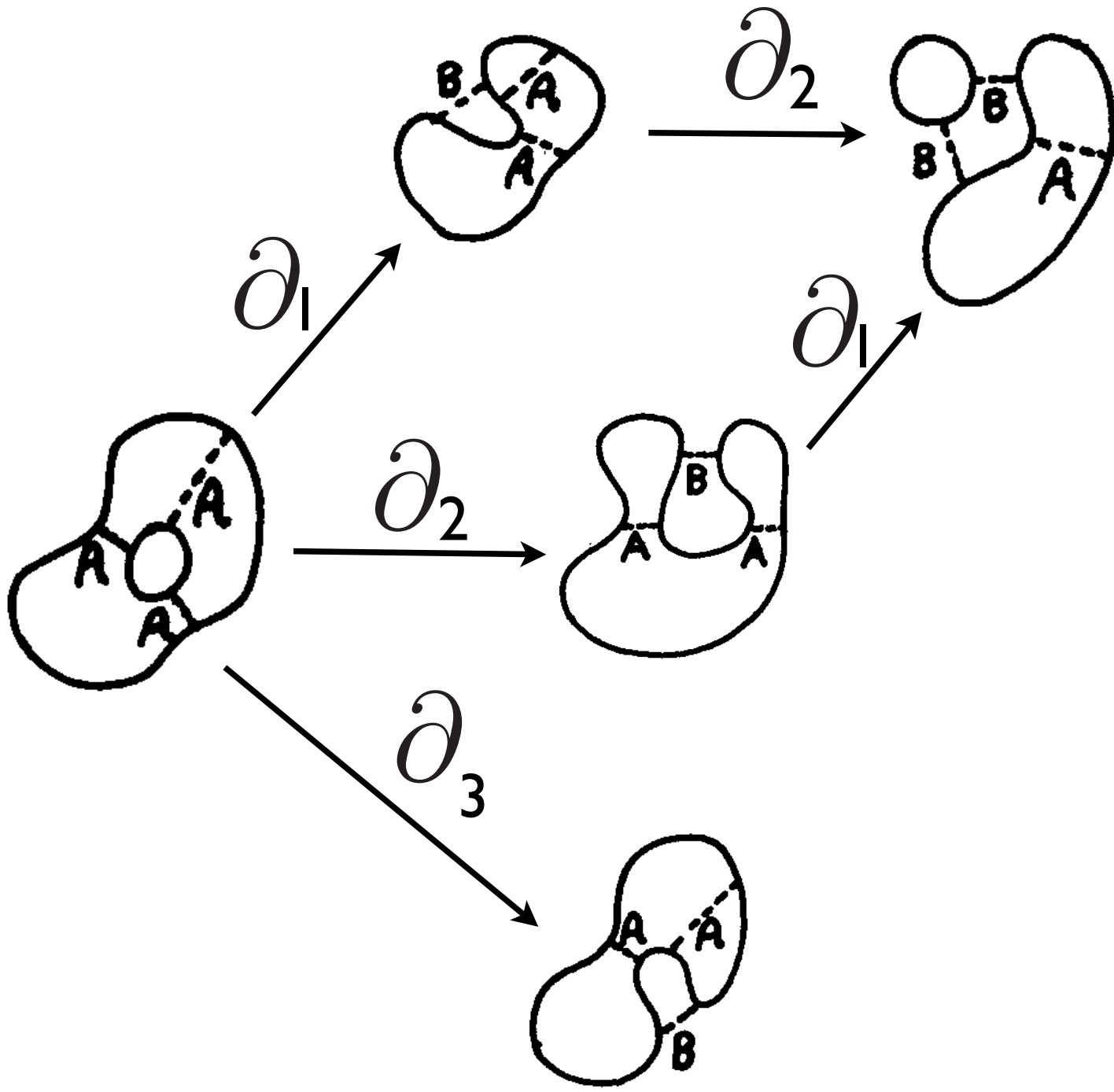


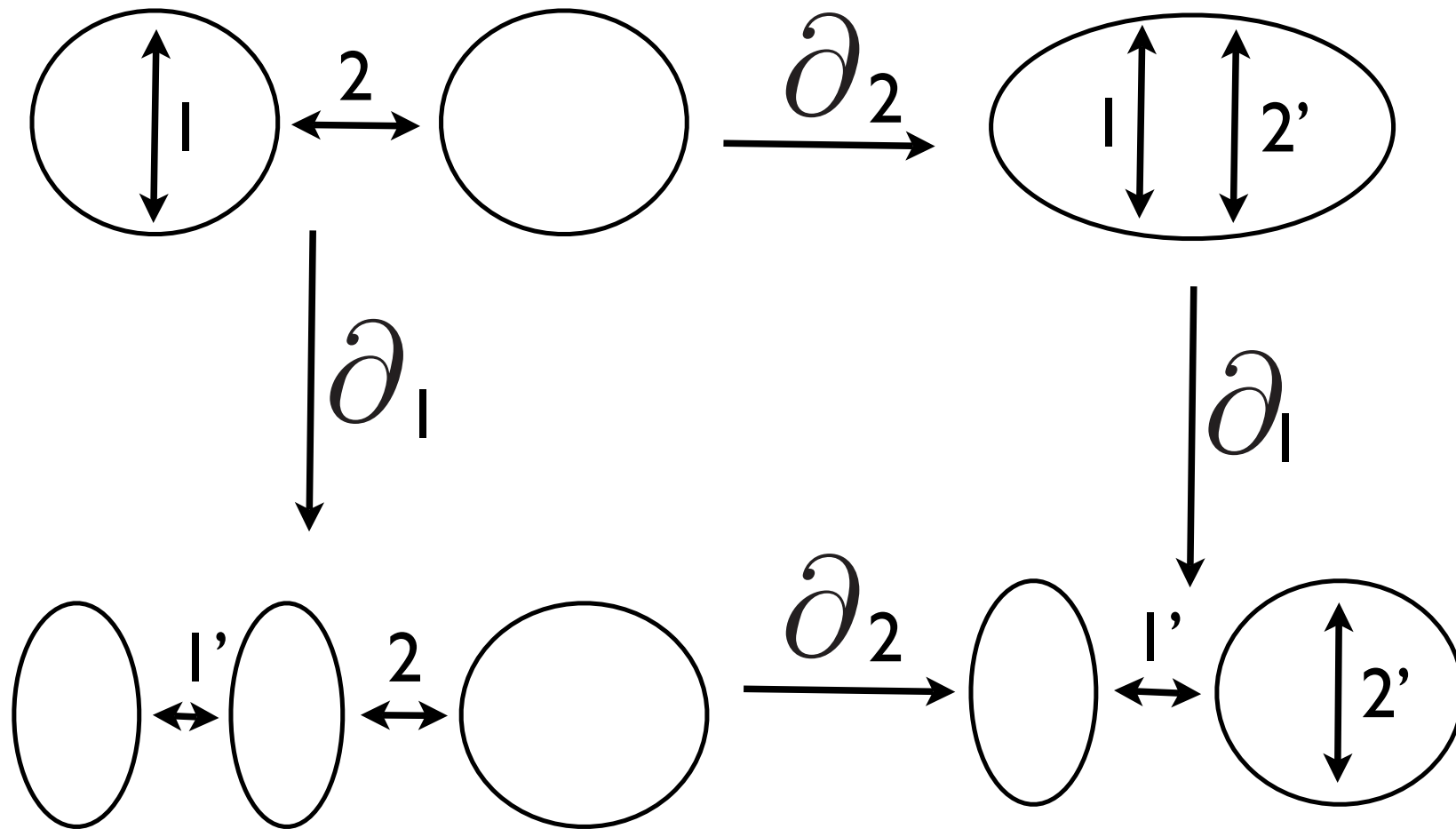
Each state loop is a module.

A collection of state loops corresponds to a tensor product of these modules.

The Functor from the cubical category to the module category demands multiplication and comultiplication in the module.

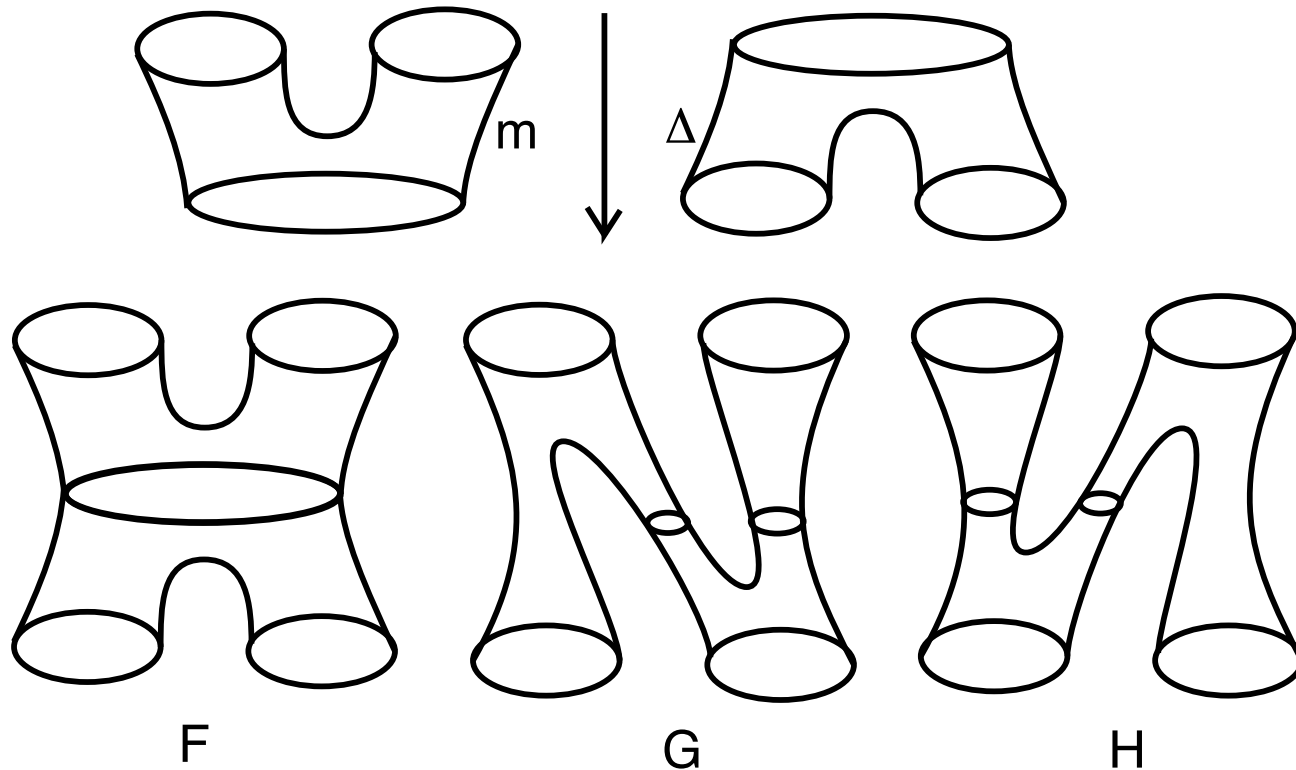






For  $d^2 = 0$ , want partial boundaries to commute.

The commutation of the partial boundaries leads to a structure of Frobenius algebra for the algebra associated to a state circle.

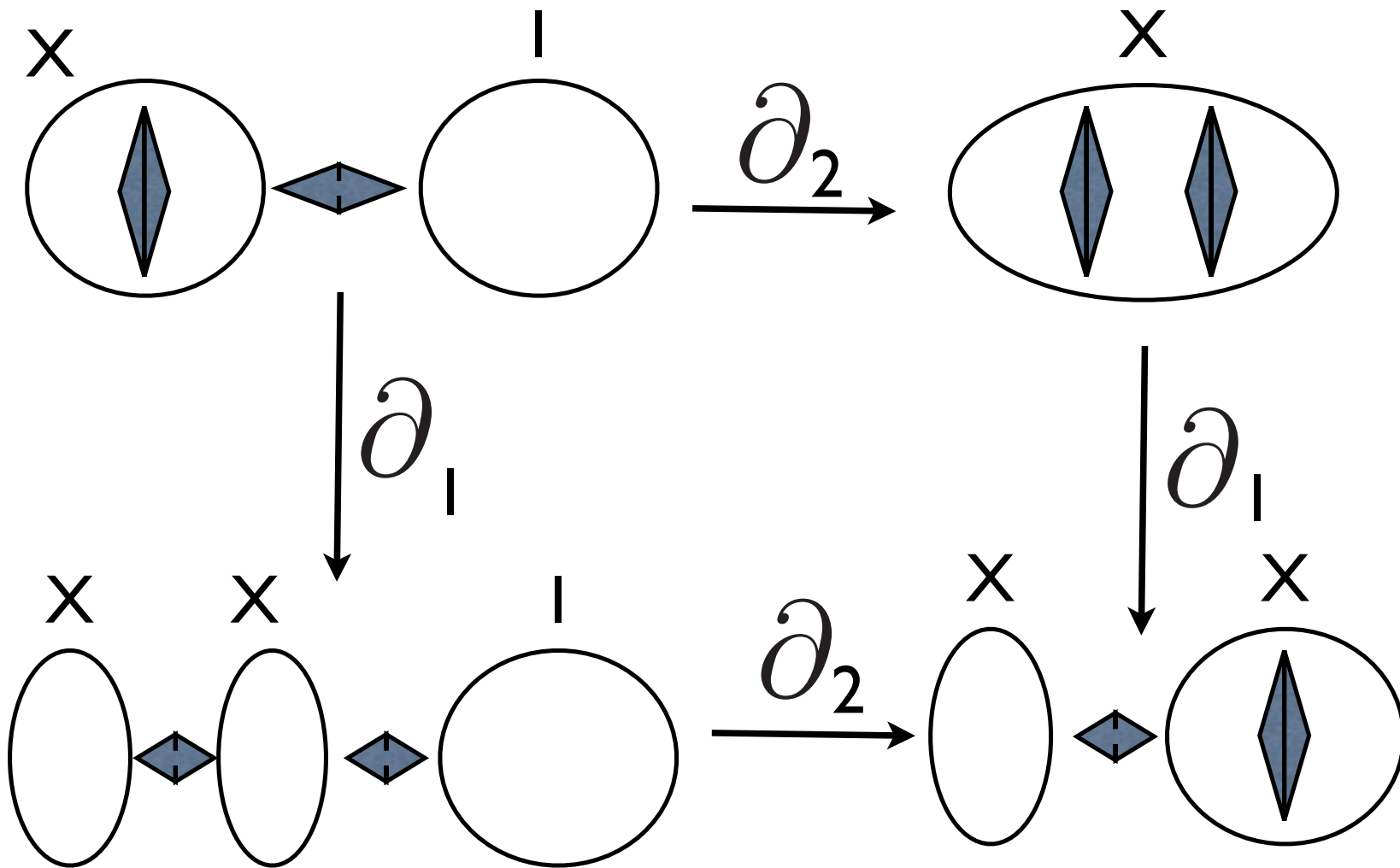


It turns out that one can take the algebra  
generated by  $1$  and  $X$  with  
 $X^2 = 0$  and

$$\Delta(X) = X \otimes X \text{ and } \Delta(1) = 1 \otimes X + X \otimes 1.$$

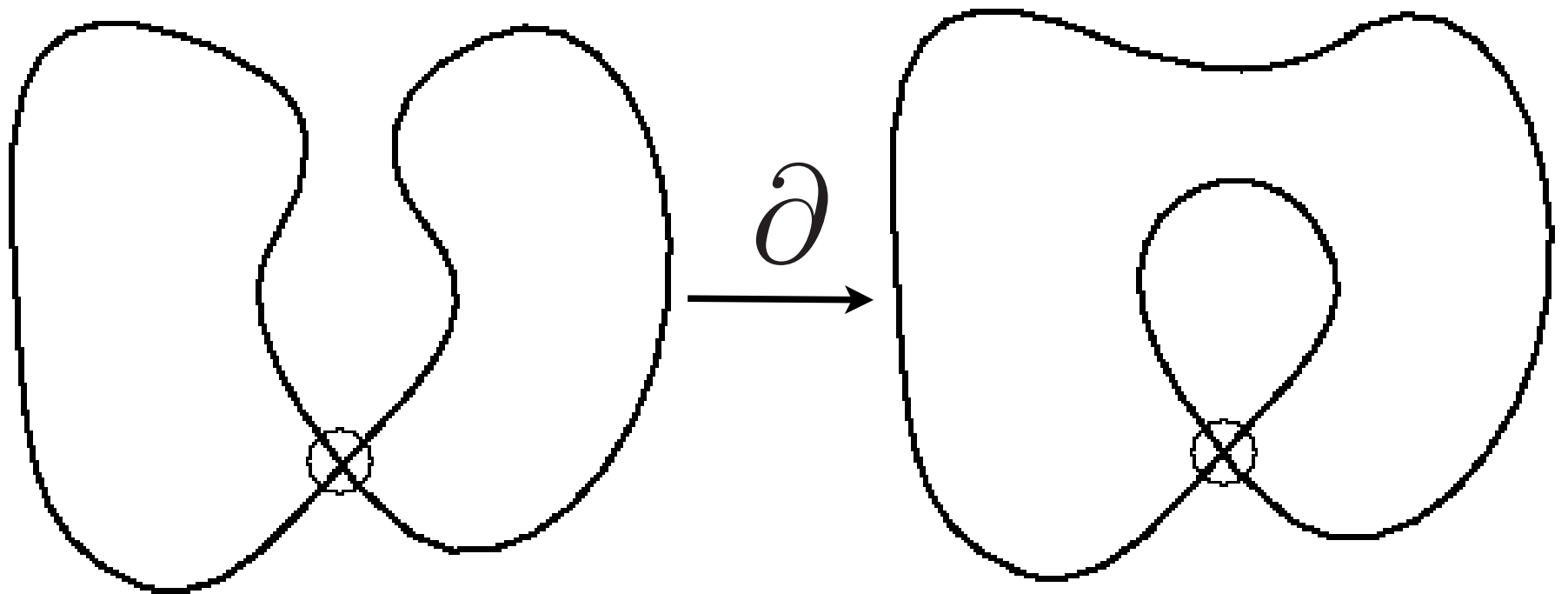
The chain complex is then generated by  
enhanced states with loop labels  $1$  and  $X$ .

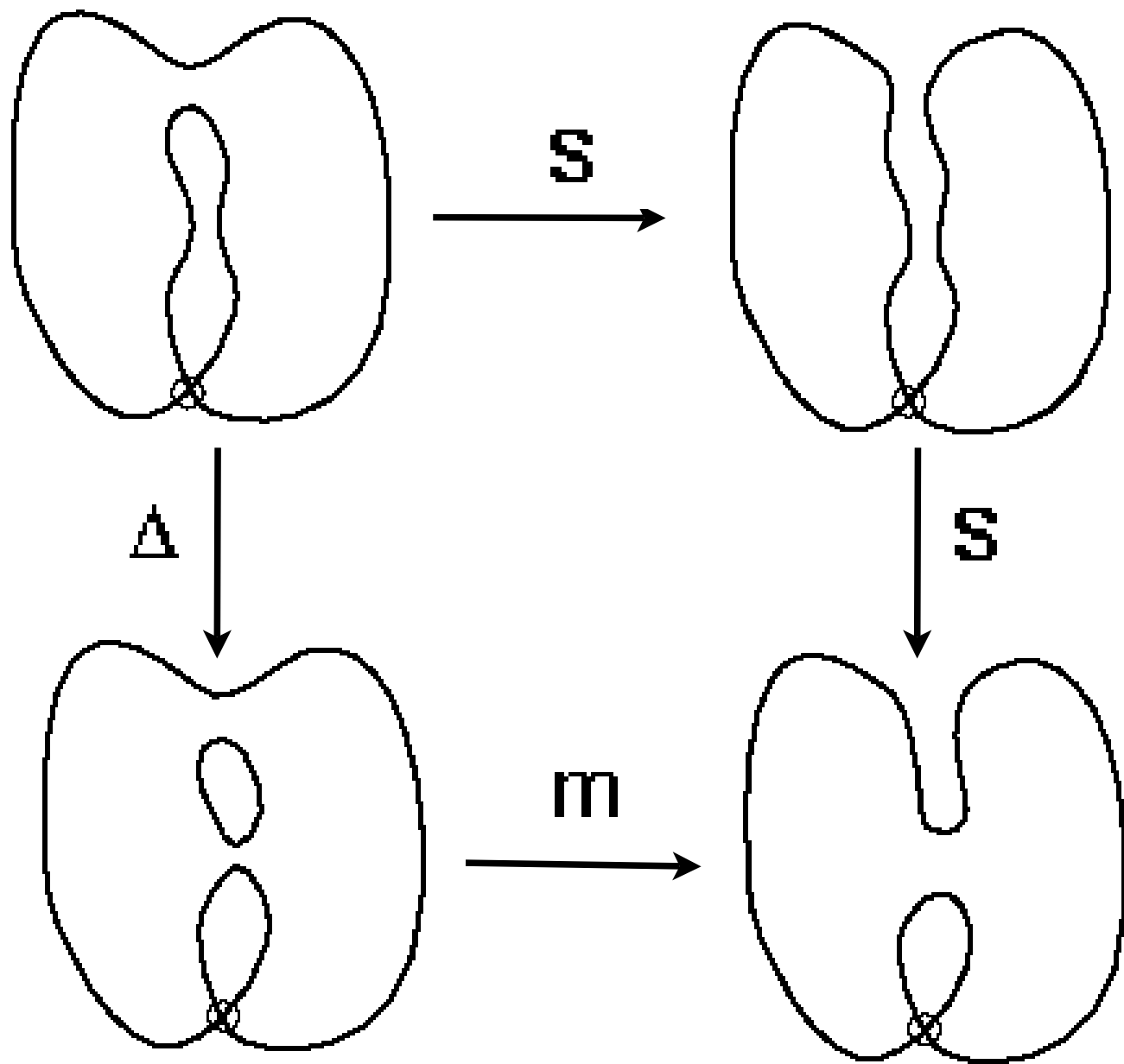




An example of the commutation of partials.

# Virtual Problem of Single Circle Morphisms





Standard Khovanov Complex works  
modulo two  
for virtual knots and links  
by setting the self -morphism to zero.

A non-trivial modification of the Khovanov  
Complex due to Vassily Manturov  
solves the problem for integral homology.

We will not discuss the use of Manturov's idea  
in this talk, but it enables the ideas that follow  
to work for virtual knots and links.

## Enhanced State Sum Formula for the Bracket

$$\langle K \rangle = \sum_s q^{j(s)} (-1)^{i(s)}$$

$$j(s) = n_B(s) + \lambda(s)$$

$i(s) = n_B(s)$  = number of B-smoothings in the state  $s$ .

$\lambda(s)$  = number of +1 loops minus number of -1 loops.  
(X)

$$\langle K \rangle = \sum_{i,j} (-1)^i q^j \dim(\mathcal{C}^{ij})$$

$\mathcal{C}^{ij}$  = module generated by enhanced states with  $i = n_B$  and  $j$  as above.

$$\langle K \rangle = \sum_{i,j} (-1)^i q^j \dim(\mathcal{C}^{ij})$$

Khovanov constructs differential acting in the form

$$\partial : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1 j}$$

For  $j$  to be constant as  $i$  increases by 1, we need

$\lambda(s)$  to decrease by 1.

[go back two slides]

The differential increases the homological grading  $i$  by 1 and leaves fixed the quantum grading  $j$ .

Then

$$\langle K \rangle = \sum_j q^j \sum_i (-1)^i \dim(\mathcal{C}^{ij}) = \sum_j q^j \chi(\mathcal{C}^{\bullet j})$$

$$\chi(H(\mathcal{C}^{\bullet j})) = \chi(\mathcal{C}^{\bullet j})$$

$$\langle K \rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet j}))$$

$$\langle K \rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet j}))$$



## RECALLING:

# A Quantum Statistical Model for Khovanov Homology and the Bracket Polynomial.

Let  $\mathcal{C}(K)$  denote a Hilbert space with basis  $|s\rangle$  where  $s$  runs over the enhanced states of a knot or link diagram  $K$ .

We define a unitary transformation.

$$U : \mathcal{C}(K) \longrightarrow \mathcal{C}(K)$$

$$U|s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle$$

$q$  is chosen on the unit circle in the complex plane.

$$|\psi\rangle = \sum_s |s\rangle$$

**Lemma.** The evaluation of the bracket polynomial is given by the following formula

$$\langle K \rangle = \langle \psi | U | \psi \rangle.$$

This gives a new quantum algorithm for the Jones polynomial (via Hadamard Test).

With  $U|s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle,$

$$\partial : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1j}$$

$$U\partial + \partial U = 0.$$

This means that the unitary transformation  
U acts on the homology so that

$$U: H(\mathcal{C}(K)) \longrightarrow H(\mathcal{C}(K))$$

$$\mathcal{C}^{\bullet,j} = \bigoplus_i \mathcal{C}^{i,j}$$

$$\langle K \rangle = \sum_s q^{j(s)} (-1)^{i(s)} = \sum_j q^j \sum_i (-1)^i \dim(\mathcal{C}^{i,j})$$

$$= \sum_j q^j \chi(\mathcal{C}^{\bullet,j}) = \sum_j q^j \chi(H(\mathcal{C}^{\bullet,j})).$$

This shows how  $\langle K \rangle$  as a quantum amplitude contains information about the homology.

## Eigenspace Picture

$$\mathcal{C}^0 = \bigoplus_{\lambda} \mathcal{C}_{\lambda}^0$$

$$\mathcal{C}_{\lambda}^{\bullet} : \mathcal{C}_{\lambda}^0 \longrightarrow \mathcal{C}_{-\lambda}^1 \longrightarrow \mathcal{C}_{+\lambda}^2 \longrightarrow \cdots \mathcal{C}_{(-1)^n \lambda}^n$$

$$\mathcal{C} = \bigoplus_{\lambda} \mathcal{C}_{\lambda}^{\bullet}$$

$$\langle \psi | U | \psi \rangle = \sum_{\lambda} \lambda \chi(H(\mathcal{C}_{\lambda}^{\bullet}))$$

## SUMMARY

We have interpreted the bracket polynomial as a quantum amplitude by making a Hilbert space  $C(K)$  whose basis is the collection of enhanced states of the bracket.

This space  $C(K)$  is naturally interpreted as the chain space for the Khovanov homology associated with the bracket polynomial.

$$\langle K \rangle = \langle \psi | U | \psi \rangle.$$

The homology and the unitary transformation  $U$  speak to one another via the formula

$$U\partial + \partial U = 0.$$

## Questions

We have shown how Khovanov homology fits into the context of quantum information related to the Jones polynomial and how the polynomial is replaced in this context by a unitary transformation  $U$  on the Hilbert space of the model. This transformation  $U$  acts on the homology, and its eigenspaces give a natural decomposition of the homology that is related to the quantum amplitude corresponding to the Jones polynomial.

The states of the model are intensely combinatorial, related to the representation of the knot or link.

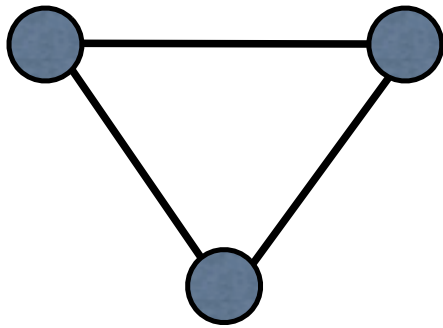
How can this formulation be used in quantum information theory and in statistical mechanics?!

# The Dichromatic Polynomial and the Potts Model

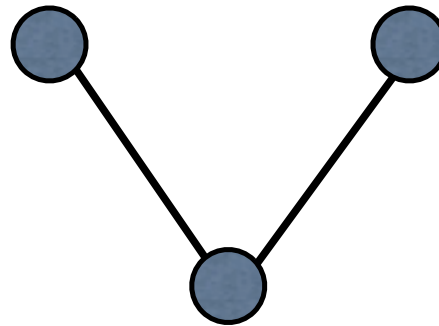
## Dichromatic Polynomial

$$Z[G](v, Q) = Z[G'](v, Q) + vZ[G''](v, Q)$$

$$Z[\bullet \sqcup G] = QZ[G].$$

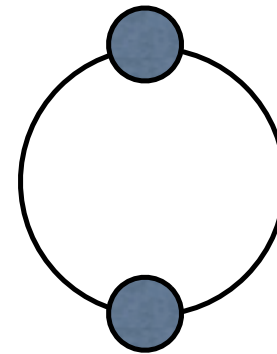


**G**



**G'**

Delete



**G''**

Contract



Precursor:

$$Z[\text{X}] = Z[\text{delete}] + v Z[\text{contract}].$$

$$Z[R \sqcup K] = Q Z[K].$$

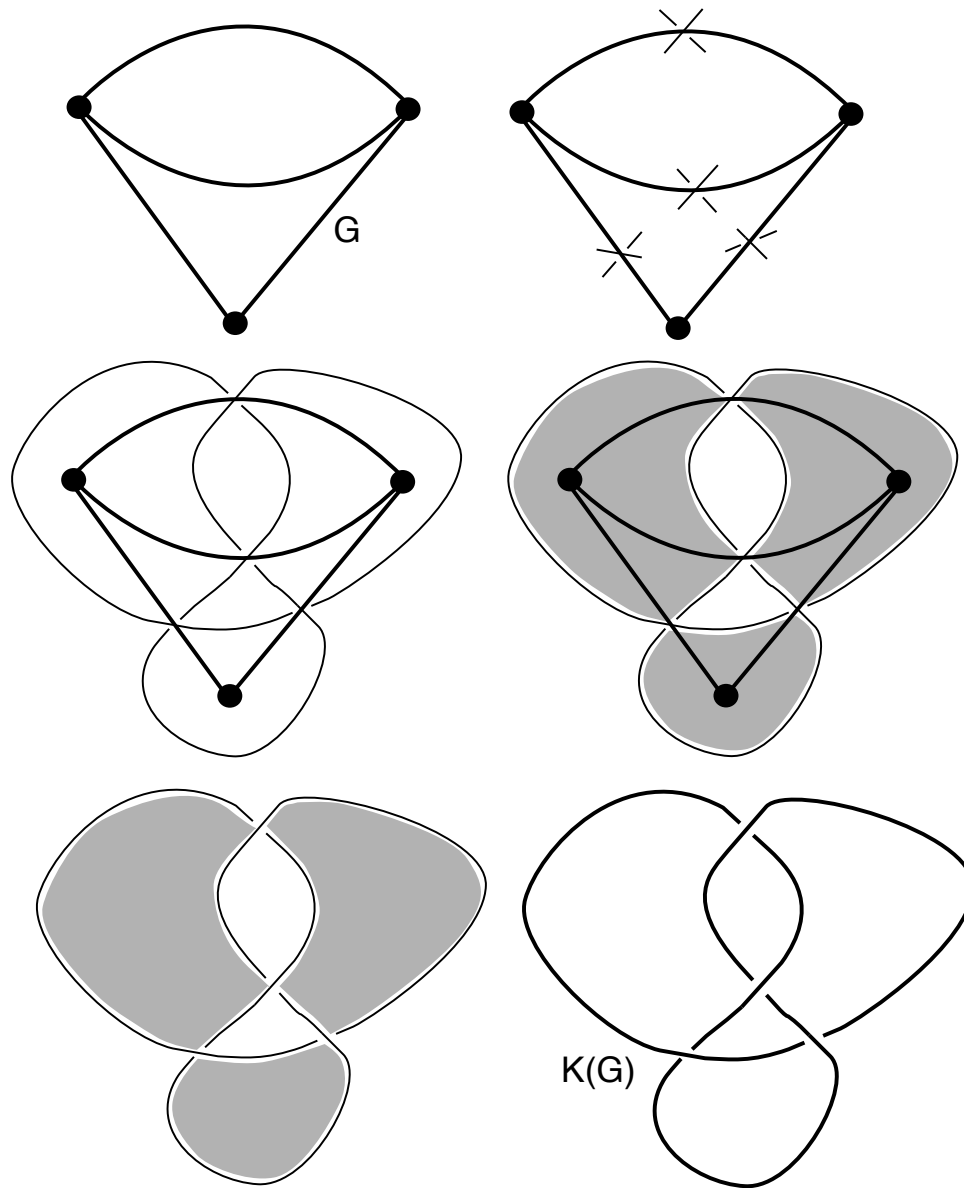


Figure 4: **Medial Graph, Checkerboard Graph and  $K(G)$**

## Partition Function

Recall that the partition function of a physical system has the form of the sum over all states  $s$  of the system the quantity

$$\sum \exp[(J/kT)E(s)]$$

where

$J = +J$  or  $-J$  (ferromagnetic or antiferromagnetic models)

$k =$  Boltzmann's constant

$T =$  Temperature

$E(s) =$  energy of the state  $s$

## Potts Model

In the Potts model, one has a graph  $G$  and assigns labels (spins, charges) to each node of the graph from a label set  $\{1, 2, \dots, Q\}$ .

A state  $s$  is such a labeling.

The energy  $E(s)$  is equal to the number of edges in the graph where the endpoints of the edge receive the same label.

For  $Q = 2$ , the Potts model is equivalent to the Ising model. The Ising model was shown by Onsager to have a phase transition in the limit of square planar lattices ( in the the 1940's).

The partition function  $P_G(Q, T)$  for the Q-state Potts model on a graph G is given by the dichromatic polynomial

$$Z[G](v, Q)$$

where

$$v = e^{J \frac{1}{kT}} - 1$$

$J = +1$  or  $-1$  (ferromagnetic or antiferromagnetic models)

$k =$  Boltzmann's constant

$T =$  Temperature

$$\text{Let } K = J \frac{1}{kT}$$

$$v = e^K - 1$$

$$P_G(Q, T) = Z[G](e^K - 1, Q)$$

$$P_G(Q, T) = \sum_{\sigma} e^{KE(\sigma)}$$

For planar graphs **G** we have

$$P_G(Q, T) = Q^{N/2} \{K(G)\}(Q, v = e^K - 1)$$

**N** = number of nodes of **G**.

**Theorem:**  $Z[G](v, Q) = Q^{N/2} \{K(G)\}$

where  $K(G)$  is an alternating link associated with the medial graph of  $G$  and

$$\left\{ \begin{array}{c} \diagdown \\ \diagup \end{array} \right\} = \left\{ \begin{array}{c} \frown \\ \smile \end{array} \right\} + Q^{-\frac{1}{2}} v \left\{ \begin{array}{c} \rangle \\ \langle \end{array} \right\}$$

$$\{ \bigcirc \} = Q^{\frac{1}{2}}$$

For the Potts Model the critical point is at

$$Q^{-\frac{1}{2}} v = 1$$

To analyze Khovanov homology, we adopt a new bracket

$$[\text{X}] = [\text{Y}] - q\rho [\text{Z}]$$

$$[\text{O}] = q + q^{-1}$$

When  $\rho = 1$ , we have the topological bracket in Khovanov form.

When

$$-q\rho = Q^{-\frac{1}{2}}v$$

$$q + q^{-1} = Q^{\frac{1}{2}}$$

we have the Potts model.



$$[K] = \sum_s (-\rho)^{n_B(s)} q^{j(s)}$$

$$[K] = \sum_{i,j} (-\rho)^i q^j \dim(\mathcal{C}^{ij})$$

$$= \sum_i q^j \sum_i (-\rho)^i \dim(\mathcal{C}^{ij}) = \sum_i q^j \chi_\rho(\mathcal{C}^{\bullet j})$$

where

$$\chi_\rho(\mathcal{C}^{\bullet j}) = \sum_i (-\rho)^i \dim(\mathcal{C}^{ij}).$$

$$[K](q, \rho = 1) = \sum_j q^j \chi_\rho(\mathcal{C}^{\bullet j}) = \sum_j q^j \chi(H(\mathcal{C}^{\bullet j}))$$

Away from  $\rho=1$ , one can ask what is the influence of the Khovanov homology on the coefficients in the expansion of

$$[K](q, \rho)$$

and corresponding questions about the Potts model.

## Tracking Potts

$$-q\rho = Q^{-\frac{1}{2}}v$$

$$q + q^{-1} = Q^{\frac{1}{2}}$$

whence

$$q^2 - \sqrt{Q}q + 1 = 0.$$

$$q = \frac{\sqrt{Q} \pm \sqrt{Q - 4}}{2}$$

At criticality Potts meets Khovanov at four colors  
and imaginary temperature!

$$\frac{1}{q} = \frac{\sqrt{Q} \mp \sqrt{Q-4}}{2}$$

Criticality:  $-\rho q = 1$

$$\rho = -\frac{1}{q} = \frac{-\sqrt{Q} \pm \sqrt{Q-4}}{2}$$

Suppose that  $\rho = 1$ .

Then  $2 = -\sqrt{Q} \pm \sqrt{Q-4}$ .

So  $4-Q = \mp \sqrt{Q} \sqrt{Q-4}$ .

And need  $Q = 4$  and  $e^K = -1$ .

Now consider  $\rho = 1$  without insisting on criticality.

$$1 = -v / (q\sqrt{Q})$$

$$\rho = -\frac{v}{\sqrt{Q}q} = v \left( \frac{-1 \pm \sqrt{1 - 4/Q}}{2} \right)$$

$$v = -q\sqrt{Q} = \frac{-Q \mp \sqrt{Q}\sqrt{Q-4}}{2}.$$

$$e^K = 1 + v = \frac{2 - Q \mp \sqrt{Q}\sqrt{Q-4}}{2}.$$

For  $Q = 2$  we have  $e^K = \pm i$ .

For  $Q = 3$ ,  $e^K = \frac{-1 \pm \sqrt{3}i}{2}$ .

For  $Q = 4$  we have  $e^K = -1$ .

For  $Q > 4$ ,  $e^K$  is real and negative.

Thus we get complex temperature values in all cases where the coefficients of the Potts model are given directly in terms of Euler characteristics from Khovanov homology.