Scattering and Generation Properties on a Nonlinear Layer and Eigen-Modes of the Linearized Problems

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Abstract – Layered structures with both negative and positive values of the coefficient of the cubic susceptibility of the nonlinear medium are investigated. The considered layers have different properties. Nonlinear layers with a negative value of the cubic susceptibility show decanalizing properties, whereas layers with a positive value of the cubic susceptibility show canalizing properties. The investigations were restricted to the third-harmonic generation. This paper presents the results of the numerical analysis characterizing the scattering and generation properties of the considered structures, taking into account the effect of weak fields at multiple frequencies.

Introduction

In the frequency domain, the resonant scattering and generation properties of nonlinear structures are determined by the proximity of the excitation frequencies of the nonlinear structures to the complex eigen-frequencies of the corresponding homogeneous linear spectral problems with the induced nonlinear dielectric permeability of the medium. The analytical continuation of these linear problems into the region of complex values of the frequency parameter allowed us to switch to the analysis of spectral problems [1]-[4]. We obtained a variety of numerical results that describe valuable properties of the nonlinear permittivity of the layers as well as their scattering and generation characteristics. By introducing a variable that describes the ratio of the *Q*-factor of eigen-oscillations at the excitation and generation frequencies, we show the following. For both canalizing and decanalizing nonlinear layers, an increase of the generated energy in the higher harmonics is accompanied by a monotonic decrease of the relative *Q*-factor of the eigen-oscillations.

Technique

The problem of resonant scattering and generation of harmonic oscillations by a nonlinear, nonmagnetic, isotropic, linearly E-polarized $\mathbf{E}=\left(E_x,0,0\right)^{\mathsf{T}}$, $\mathbf{H}=\left(0,H_y,H_z\right)^{\mathsf{T}}$, cubically polarizable $\mathbf{P}^{(\mathrm{NL})}=\left(P_x^{(\mathrm{NL})},0,0\right)^{\mathsf{T}}$, layered dielectric structure is investigated in a self-consistent formulation (see Fig. 1). The time dependency has the form $\exp(-in\omega t)$, $n=1,2,\ldots$

The variables x,y,z,t denote dimensionless spatial-temporal coordinates such that the thickness of the layer is equal to $4\pi\delta$; $n\kappa=n\omega/c=2\pi/\lambda_{n\kappa}$ are dimensionless frequencies; $\lambda_{n\kappa}$ denote the lengths of the incident waves; $\omega=\kappa c$ is the dimensionless circular frequency and c is a dimensionless parameter, the absolute value of which is equal to the velocity of light within the medium containing the layer, $\mathrm{Im}\,c=0$.

The incidence of a packet of plane waves onto the layer at the angles $\left\{\varphi_{n\kappa},\pi-\varphi_{n\kappa}:\left|\varphi_{n\kappa}\right|<\pi/2\right\}_{n=1}^{3}$ and with respect to the amplitudes $\left\{a_{n\kappa}^{\mathrm{inc}},b_{n\kappa}^{\mathrm{inc}}\right\}_{n=1}^{3}$ at the

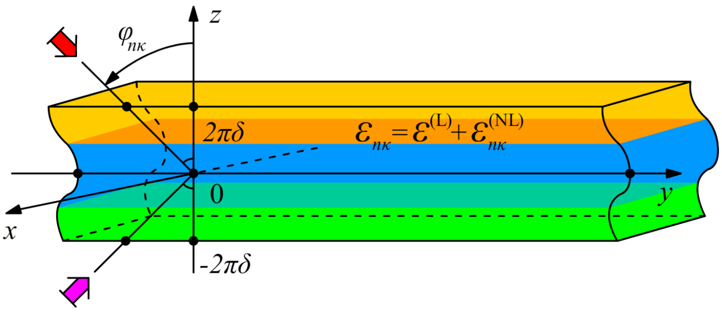


Fig. 1. The nonlinear layered dielectric structure.

frequencies $\{n\kappa\}_{n=1}^{3}$ is considered, where the excitation field consists of a strong field at the frequency κ (generating a field at the triple frequency) and of weak fields at the frequencies 2κ and 3κ (influencing on the process of generation of the third harmonic).

In such a situation, taking into account Kleinman‘s rule (i.e. the equality of all the susceptibility tensor components $\chi^{(3)}_{xxxx}$ at the multiple frequencies [5]), the problem under consideration can be described by a system of nonlinear boundary value problems [1]-[3]

$$\left[\nabla^2 + (n\kappa)^2 \varepsilon_{n\kappa}(z,\alpha(z),E_x(\kappa;y,z),E_x(2\kappa;y,z),E_x(3\kappa;y,z))\right] E_x(n\kappa;y,z) = \\ -(n\kappa)^2 \alpha(z)\left[\delta_{n1} E_x^{\,2}(2\kappa;y,z) E_x^{\,*}(3\kappa;y,z) + \delta_{n3}\left\{\tfrac{1}{3} E_x^{\,3}(\kappa;y,z) + E_x^{\,2}(2\kappa;y,z) E_x^{\,*}(\kappa;y,z)\right\}\right],\ n = 1, 2, 3, \quad (1)$$

together with the following generalized boundary conditions:

(C1) $E_x(n\kappa;y,z) = U(n\kappa;z)\exp(i\Phi_{n\kappa}y)$, the quasi-homogeneity condition w.r.t. y,

(C2) $\Phi_{n\kappa} = n\Phi_\kappa$ or $\varphi_{n\kappa} = \varphi_\kappa$, the condition of phase synchronism of waves [2],

(C3) $\mathbf{E}_{\text{tg}}(n\kappa;y,z)$ and $\mathbf{H}_{\text{tg}}(n\kappa;y,z)$ are continuous at the boundary layers $\varepsilon_{n\kappa}$,

(C4) $E_x^{\text{scat/gen}}(n\kappa;y,z) = \begin{Bmatrix} a_{n\kappa}^{\text{scat/gen}} \\ b_{n\kappa}^{\text{scat/gen}} \end{Bmatrix} \exp\left[i\left(\Phi_{n\kappa}y \pm \Gamma_{n\kappa}(z \mp 2\pi\delta)\right)\right],\ z \gtrless \pm 2\pi\delta$, for $\operatorname{Im}\Gamma_{n\kappa} \equiv 0$ and $\operatorname{Re}\Gamma_{n\kappa} > 0$, the radiation condition w.r.t. the scattered and generated fields.

Here: $\nabla^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2$, δ_n^k – Kronecker’s symbol, $\mathbf{E}_{\text{tg}}(n\kappa;y,z)$ and $\mathbf{H}_{\text{tg}}(n\kappa;y,z)$ – the tangential components of the intensity vectors of the full electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$, $\Gamma_{n\kappa} = \sqrt{(n\kappa)^2 - \Phi_{n\kappa}^{\,2}}$ and $\Phi_{n\kappa} = n\kappa\sin(\varphi_{n\kappa})$ – the transverse and longitudinal propagation constants of the nonlinear structure, $\varepsilon_{n\kappa} = \left\{1, |z| > 2\pi\delta;\ \text{and}\ \varepsilon^{(L)} + \varepsilon^{(NL)}_{n\kappa},\ |z| \le 2\pi\delta\right\}$,

$$\varepsilon^{(NL)}_{n\kappa} = \alpha(z)\left[\sum_{m=1}^{3}|E_x(m\kappa;y,z)|^2 + \left\{\delta_{n1}\frac{\left[E_x^{\,*}(\kappa;y,z)\right]^2}{E_x(\kappa;y,z)} + \delta_{n2}\frac{E_x^{\,*}(2\kappa;y,z)}{E_x(2\kappa;y,z)}E_x(\kappa;y,z)\right\}E_x(3\kappa;y,z)\right],$$

$\varepsilon^{(L)} = 1 + 4\pi\chi^{(1)}_{xx}(z)$, $\alpha(z) = 3\pi\chi^{(3)}_{xxxx}(z)$ – the function of cubic susceptibility of the nonlinear medium, $\chi^{(1)}_{xx}$ and $\chi^{(3)}_{xxxx}$ – components of the susceptibility tensors of the nonlinear medium.

The sought complex Fourier amplitudes of the total scattered and generated fields in the problem (1), (C1)-(C4) at the multiple frequencies $\{n\kappa\}_{n=1}^{3}$ can be represented in the form

$$E_x(n\kappa; y,z) = U(n\kappa;z)\exp(i\Phi_{n\kappa}y) = \begin{cases} a_{n\kappa}^{\text{inc}}\exp\left(i\left(\Phi_{n\kappa}y - \Gamma_{n\kappa}(z-2\pi\delta)\right)\right) + a_{n\kappa}^{\text{scat/gen}}\exp\left(i\left(\Phi_{n\kappa}y + \Gamma_{n\kappa}(z-2\pi\delta)\right)\right), & z > 2\pi\delta, \\ U(n\kappa;z)\exp(i\Phi_{n\kappa}y), & |z| \le 2\pi\delta, \\ b_{n\kappa}^{\text{inc}}\exp\left(i\left(\Phi_{n\kappa}y + \Gamma_{n\kappa}(z+2\pi\delta)\right)\right) + b_{n\kappa}^{\text{scat/gen}}\exp\left(i\left(\Phi_{n\kappa}y - \Gamma_{n\kappa}(z+2\pi\delta)\right)\right), & z < -2\pi\delta. \end{cases} \quad (2)$$

Taking into consideration (2), the nonlinear system (1), (C1)-(C4) is equivalent to a system (see [1-3]) of nonlinear boundary-value problems of Sturm-Liouville type

$$\begin{aligned} &\left[d^2/dz^2 + \Gamma_{n\kappa}^2 - (n\kappa)^2\{1-\varepsilon_{n\kappa}(z,\alpha(z),U(\kappa;z),U(2\kappa;z),U(3\kappa;z))\}\right]U(n\kappa;z) = \\ &\quad -(n\kappa)^2\alpha(z)\left(\delta_{n1}U^2(2\kappa;z)U^*(3\kappa;z) + \delta_{n3}\left\{\tfrac{1}{3}U^3(\kappa;z) + U^2(2\kappa;z)U^*(\kappa;z)\right\}\right), \quad |z| \le 2\pi\delta, \\ &\left[i\Gamma_{n\kappa} - d/dz\right]U(n\kappa;2\pi\delta) = 2i\Gamma_{n\kappa}\overline{U}^{\text{inc}}(n\kappa;2\pi\delta), \\ &\left[i\Gamma_{n\kappa} + d/dz\right]U(n\kappa;-2\pi\delta) = 2i\Gamma_{n\kappa}\underline{U}^{\text{inc}}(n\kappa;-2\pi\delta), \quad n = 1,2,3, \end{aligned} \quad (3)$$

and also to a system of one-dimensional nonlinear integral equations w.r.t. the unknown functions $U(n\kappa;\cdot) \in L_2(-2\pi\delta, 2\pi\delta)$,

$$\begin{aligned} &U(n\kappa;z) + \frac{i(n\kappa)^2}{2\Gamma_{n\kappa}}\int_{-2\pi\delta}^{2\pi\delta}\exp(i\Gamma_{n\kappa}|z-\xi|)\left[1-\varepsilon_{n\kappa}(\xi,\alpha(\xi),U(\kappa;\xi),U(2\kappa;\xi),U(3\kappa;\xi))\right]U(n\kappa;\xi)d\xi = \\ &\frac{i(n\kappa)^2}{2\Gamma_{n\kappa}}\int_{-2\pi\delta}^{2\pi\delta}\exp(i\Gamma_{n\kappa}|z-\xi|)\alpha(\xi)\left[\delta_{n1}U^2(2\kappa;\xi)U^*(3\kappa;\xi) + \delta_{n3}\left\{\tfrac{1}{3}U^3(\kappa;\xi) + U^2(2\kappa;\xi)U^*(\kappa;\xi)\right\}\right]d\xi + \\ &\overline{U}^{\text{inc}}(n\kappa;z) + \underline{U}^{\text{inc}}(n\kappa;z), \quad n = 1,2,3. \end{aligned} \quad (4)$$

Here $\overline{U}^{\text{inc}}(n\kappa;z) = a_{n\kappa}^{\text{inc}}\exp[-i\Gamma_{n\kappa}(z-2\pi\delta)]$, $\underline{U}^{\text{inc}}(n\kappa;z) = b_{n\kappa}^{\text{inc}}\exp[+i\Gamma_{n\kappa}(z+2\pi\delta)]$, $n = 1,2,3$.

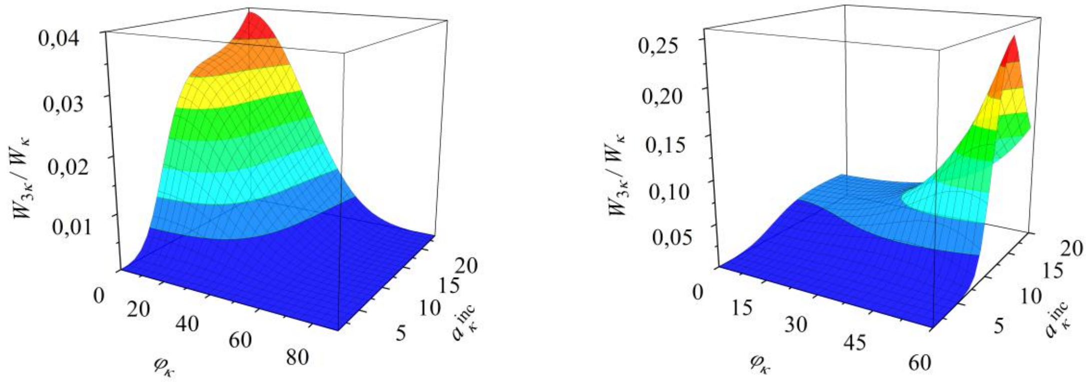
The solution of the problem (1), (C1)-(C4), represented in (2), can be obtained from (3) or (4) using the formulas $U(n\kappa;2\pi\delta) = a_{n\kappa}^{\text{inc}} + a_{n\kappa}^{\text{scat/gen}}$, $U(n\kappa;-2\pi\delta) = b_{n\kappa}^{\text{inc}} + b_{n\kappa}^{\text{scat/gen}}$, $n = 1,2,3$.

According to [2-3], the application of suitable quadrature rules to the system (4) leads to a system of complex-valued nonlinear algebraic equations of the second kind

$$\left[\mathbf{I} - \mathbf{B}_{n\kappa}(\mathbf{U}_\kappa, \mathbf{U}_{2\kappa}, \mathbf{U}_{3\kappa})\right]\mathbf{U}_{n\kappa} = \delta_{n1}\mathbf{C}_\kappa(\mathbf{U}_{2\kappa}, \mathbf{U}_{3\kappa}) + \delta_{n3}\mathbf{C}_{3\kappa}(\mathbf{U}_\kappa, \mathbf{U}_{2\kappa}) + \overline{\mathbf{U}}_{n\kappa}^{\text{inc}} + \underline{\mathbf{U}}_{n\kappa}^{\text{inc}}, \; n = 1,2,3 \quad (5)$$

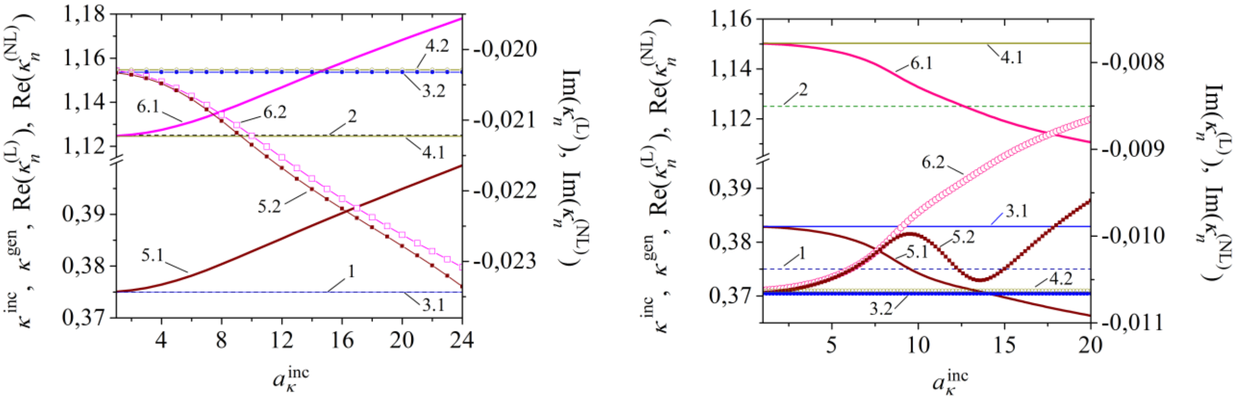
where $\mathbf{U}_{n\kappa} = \{U_l(n\kappa)\}_{l=1}^N \approx \{U(n\kappa;z_l)\}_{l=1}^N$ – the vectors of the unknown approximate values of the solution, $\left\{\{z_l\}_{l=1}^N : z_1 = -2\pi\delta < ... < z_l < ... < z_N = 2\pi\delta\right\}$ – a discrete set on interpolation nodes, $\mathbf{I} = \{\delta_{lm}\}_{l,m=1}^N$ – the identity matrix, $\mathbf{B}_{n\kappa}(\mathbf{U}_\kappa, \mathbf{U}_{2\kappa}, \mathbf{U}_{3\kappa})$ – nonlinear matrices, $\mathbf{C}_\kappa(\mathbf{U}_{2\kappa}, \mathbf{U}_{3\kappa})$, $\mathbf{C}_{3\kappa}(\mathbf{U}_\kappa, \mathbf{U}_{2\kappa})$ – the vectors of the right-hand sides determined by the choice of the quadrature rule and $\overline{\mathbf{U}}_{n\kappa}^{\text{inc}} = \left\{a_{n\kappa}^{\text{inc}}\exp[-i\Gamma_{n\kappa}(z_l - 2\pi\delta)]\right\}_{l=1}^N$, $\underline{\mathbf{U}}_{n\kappa}^{\text{inc}} = \left\{b_{n\kappa}^{\text{inc}}\exp[+i\Gamma_{n\kappa}(z_l + 2\pi\delta)]\right\}_{l=1}^N$ – the vectors induced by the incident wave packets. A solution of (6) can be found iteratively by the help of a block Jacobi method, where at each step a system of linearized algebraic equations is solved.

The analytic continuation of the linearized nonlinear problems into the region of complex values of the frequency parameter allows us to switch to the analysis of spectral problems [1-4]. The problem of finding the eigen-frequencies κ_n and the eigen-fields $\mathbf{U}_{\kappa_n}$ reads as follows:



(a) (b)

Fig. 2. Relative part of energy generated in the third harmonic for $\alpha = -0.01$ (a) and for $\alpha = +0.01$ (b).



(a) (b)

Fig. 3. Curves at $\varphi_\kappa = 0^0$ (a) and $\varphi_\kappa = 60^0$ (b): 1 … $\kappa = \kappa^{\mathrm{inc}} = 0.375$, 2 … $3\kappa = \kappa^{\mathrm{gen}} = 3\kappa^{\mathrm{inc}}$; 3.1 … $\mathrm{Re}\left(\kappa_1^{(\mathrm{L})}\right)$, 3.2 … $\mathrm{Im}\left(\kappa_1^{(\mathrm{L})}\right)$, 4.1 … $\mathrm{Re}\left(\kappa_3^{(\mathrm{L})}\right)$, 4.2 … $\mathrm{Im}\left(\kappa_3^{(\mathrm{L})}\right)$ for $\alpha = 0$; 5.1 … $\mathrm{Re}\left(\kappa_1^{(\mathrm{NL})}\right)$, 5.2 … $\mathrm{Im}\left(\kappa_1^{(\mathrm{NL})}\right)$, 6.1 … $\mathrm{Re}\left(\kappa_3^{(\mathrm{NL})}\right)$, 6.2 … $\mathrm{Im}\left(\kappa_3^{(\mathrm{NL})}\right)$ for $\alpha = -0.01$ (a) and for $\alpha = +0.01$ (b).

$$\begin{cases} f_{n\kappa}(\kappa_n) = \det\left[\mathbf{I} - \mathbf{B}_{n\kappa}(\kappa_n)\right] = 0, \\ \left[\mathbf{I} - \mathbf{B}_{n\kappa}(\kappa_n)\right]\mathbf{U}_{\kappa_n} = \mathbf{0}; \qquad \kappa \equiv \kappa^{\mathrm{inc}}; \quad \kappa_n \in \Omega_{n\kappa} \subset \mathrm{H}_{n\kappa}, \end{cases} \tag{6}$$

where $\kappa_n \in \Omega_{n\kappa} \subset \mathrm{H}_{n\kappa}$, at $\kappa \equiv \kappa^{\mathrm{inc}}$, $n = 1, 2, 3$, $\Omega_{n\kappa}$ are the sets of eigen-frequencies and $\mathrm{H}_{n\kappa}$ denote two-sheeted Riemann surfaces (see [2-3]).

Results

Consider a decanalizing ($\alpha(z) < 0$) and a canalizing ($\alpha(z) > 0$) nonlinear dielectric structure with the parameters $\varepsilon^{(\mathrm{L})}(z) = 16$, $\alpha(z) = \mp 0.01$, $\delta = 0.5$. The excitation of the nonlinear layer takes place from above by only one strong top electromagnetic field at the basic frequency, i.e., $\left\{a_\kappa^{\mathrm{inc}} \neq 0, a_{2\kappa}^{\mathrm{inc}} = 0, a_{3\kappa}^{\mathrm{inc}} = 0\right\}$, $\left\{b_\kappa^{\mathrm{inc}} = 0, b_{2\kappa}^{\mathrm{inc}} = 0, b_{3\kappa}^{\mathrm{inc}} = 0\right\}$ and $\kappa = \kappa^{\mathrm{inc}} = 0.375$.

We define by $W_{n\kappa} = \left|a_{n\kappa}^{\mathrm{scat/gen}}\right|^2 + \left|b_{n\kappa}^{\mathrm{scat/gen}}\right|^2$ the total energy of the scattered and generated fields at the frequencies $n\kappa$ and consider the quantity $W_{3\kappa}/W_\kappa$ which characterizes the

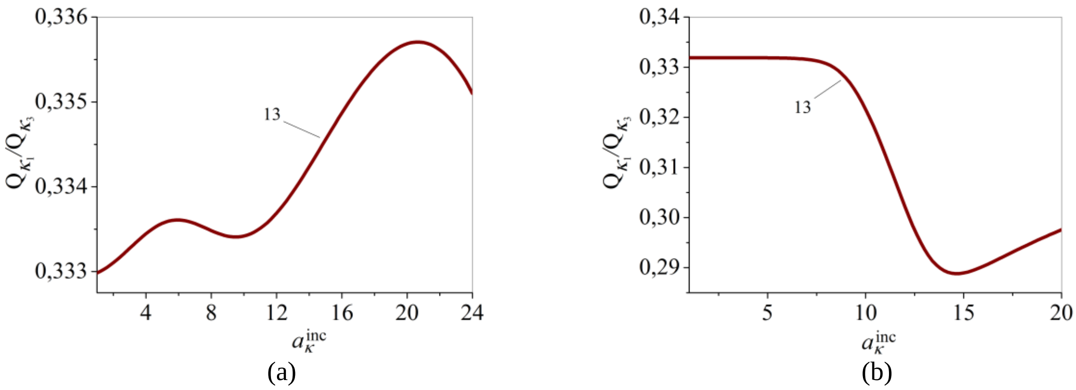


Fig. 4. The relative *Q*-factor, curves 13 … $Q_{\kappa_1}/Q_{\kappa_3}$ at $\kappa^{\rm inc} = 0.375, \kappa_n = \kappa_n^{(\rm NL)}$, $n = 1, 3$ for $\varphi_\kappa = 0^0$, $\alpha = -0.01$ (a) and for $\varphi_\kappa = 60^0$, $\alpha = +0.01$ (b).

portion of energy generated in the third harmonic in comparison to the energy scattered in the first harmonic, see Fig. 2.

Denote by $Q_{\kappa_n} = -\mathrm{Re}(\kappa_n)/[2\,\mathrm{Im}(\kappa_n)]$ the *Q*-factor of the eigen-oscillations of the spectral problem (6) at the eigen-frequencies $\kappa_n \in \Omega_{n\kappa} \subset \mathrm{H}_{n\kappa}$, see [2], [4] and Fig. 3. In the numerical experiments, the parameter $Q_{13} = Q_{\kappa_1}/Q_{\kappa_3}$ of the relative *Q*-factor of the eigen-oscillations is of particular interest, see Fig. 4. For an increasing amplitude of the exciting field, an increase of the generated energy in the higher harmonics is accompanied by a monotonic decrease of the relative *Q*-factor of the eigen-oscillations, see Figs. 3 and 4.

Conclusion

This paper presents the results of numerical computations characterizing the scattering and generation properties of the considered structures. They demonstrate the possibility of controlling the scattering and generating properties of a nonlinear structure by means of the intensities of the excitation fields. They also indicate the possibility of designing a frequency multiplier and other electrodynamic devices containing nonlinear dielectrics with controllable permittivity.

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